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FOUNDATIONS

★**Rosenbloom, Paul C. The Elements of Mathematical Logic.** Dover Publications, Inc., New York, N. Y., 1950. vi+iv+214 pp. $2.95.

Dies ist ein Buch, das für Mathematiker geschrieben ist, nicht für Philosophen. Philosophische Fragen nach der Begründung der Logik bleiben unerörtert; es werden vielmehr auf der Grundlage der gegenwärtig anerkannten Mathematik die neuartigen, rein mathematisch interessanten Probleme der logistischen Kalküle und der Syntax beliebiger Kalküle behandelt. Das Buch ist daher allen Mathematikern, die diese Gebiete kennen lernen wollen, sehr zu empfehlen. Kap. I (Klassenlogik) bringt die Axiome der Booleschen Verbände über den Grundbegriffen $\wedge$ (Konjunktion), $\sim$ (Negation), $=$ (Identität) mit Interpretation und den Beweisen der wichtigsten Sätzen. Die Darstellbarkeit jedes Booleschen Verbandes als Mengenverband und das Kriterium für Darstellbarkeit als Verband aller Untermengen einer Menge werden bewiesen. Kap. II (Aussagenlogik) beginnt mit den Wahrheitswerttabellen, aus denen folgt, dass die "Aussagen" einen Booleschen Verband bilden. Ein System mit den Grundbegriffen $\supset$ (Implikation), $\sim$ (Negation), F (Klasse der Sätze) wird als äquivalente Axiomatisierung nachgewiesen. Der Übergang von solchen Axiomensystemen zu Kalkülen (mit bedeutungsfreien Zeichen) wird ausführlich beschrieben. Jedes Modell (Interpretation) des aussagenlogischen Systems heisst eine "Logik." Eine Logik heisst kategorisch, wenn für jede Aussage p entweder p oder $\sim p$ ein Satz ist. Das ist genau dann der Fall, wenn F ein maximales sum-Ideal ist. Auch das Deduktionstheorem wird bewiesen. Für mehrwertige Logiken (Post), Modallogiken (Lewis, H. B. Smith), und die intuitionistische Logik (Heyting) werden die Axiomensysteme angeführt, und einiges über ihre Interpretation gesagt. Die Kritik des Verf. (S. 56) am Intuitionismus trifft diesen allerdings nicht. Um den "dangerous quicksand" philosophischer Kontroverse zu vermeiden, wird das schöne Prinzip "put up or shut up" empfohlen, nach dem eine Kritik, dass ein logisches System gewisse Eigenschaften haben sollte bzw. nicht haben sollte, nur dann erlaubt ist, wenn ein System vorgelegt wird das beweisbar diese Eigenschaften hat bzw. nicht hat.

Kap. III (Prädikatenlogik) geht aus von inhaltlichen Überlegungen über "Funktionen" deren Werte Aussagen sind, und über "für alle" und "für ein" (es gibt). Dann wird ein Axiomensystem L_1 der Prädikatenlogik 1. Stufe über den Grundbegriffen I (Klasse der Individuen), $f_1, \cdots, f_n$ (Klassen der 1-, $\cdots$, n-stelligen Prädikate), P (Klasse der Aussagen), F (Klasse der Sätze), $\supset$, $\sim$, $A^n\alpha_1\cdots\alpha_n$ (die Aussage, dass A^n aus f_n auf $\alpha_1, \cdots, \alpha_n$ aus I zutrifft), $\prod A^n$ (das $(n-1)$-stellige Prädikat, für das $\prod A^n\alpha_2\cdots\alpha_n$ gilt, wenn für alle x: $A^n x\alpha_2\cdots\alpha_n$) aufgestellt. Die Widersprüchlichkeit des uneingeschränkten Komprehensionsaxioms, wie die Kompliziertheit der verzweigten Typenlogik werden hervorgehoben, und es werden daher die unverzweigte Typenlogik L_2, die typenfreien Kalküle von Quine L_3 und Zermelo L_z mit den eingeschränkten Komprehensionsaxiomen als die zur Zeit besten Lösungen behandelt. Eine ausführliche Darstellung findet aber auch die kombinatorische Logik (Curry) aufgebaut auf den zwei Kombinatoren K, A, definiert durch $\|Kab = a$ und $\||Aabc = \|ac|bc$ für alle Operatoren a, b, c, in klammerfreier Schreibweise, d.h. $|ab$ für $a(b)$. Hervorzuheben ist der einfache Beweis der Vollständigkeit. Auch der Church'sche Aufbau der kombinatorischen Logik (λ-conversion) wird dargestellt und die Fortführung bis zu "real mathematics" skizziert. Die Anfänge von Mengenlehre und Arithmetik werden auch von L_3 aus entwickelt. Anschliessend werden die sog. Paradoxien des inhaltlichen Denkens (Epimenides, Russell, Richard) gestreift und dann das Auswahlaxiom in mehreren äquivalenten Formen besprochen. Verf. scheint das Axiom auf Grund der bisherigen häufigen Verwendung für gerechtfertigt zu halten.

Kap. IV (Kalkülsyntax) behandelt die moderne Entwicklung der Syntax beliebiger "Sprachen," allerdings keine Widerspruchsfreiheitsbeweise im Sinne der Hilbertschen Metamathematik. Sprachen mit endlich vielen Zeichen und endlich vielen Regeln zur Konstruktion ("production") der Sätze heissen kanonisch. Der Post'sche Satz, dass jede kanonische Sprache zu einer normalen Sprache (mit einem Axiom und einfachster Form der Regeln) erweitert werden kann, ohne die Menge der Sätze zu ändern, wird ohne Beweis formuliert. Für den Unvollständigkeitssatz von Gödel und den Unentscheidbarkeitssatz von Church werden dann aber alle wesentlichen Schritte durchgeführt. Hierbei wird die "Univeralsprache" $\mathfrak{B}_2$ benutzt, die alle Sätze aller normalen Sprachen (aus zwei Zeichen) enthält, wodurch gegenüber der Gödel'schen Arithmetisierung grosse Vereinfachungen erzielt werden. Das Buch schliesst mit wichtigen historischen und bibliographischen Hinweisen. Zu jeden § enthält das Buch ausserdem zahlreiche Übungsaufgaben, wofür dem Verf. sehr zu danken ist. *P. Lorenzen* (Bonn).

Yonemitsu, Naoto. On systems of strict implication. Tôhoku Math. J. (2) 3, 48–58 (1951).

Discussion of the system constructed by Vredenduin [J. Symbolic Logic 4, 73–76 (1939)] in order to avoid paradoxical consequences such as 19.74 and 19.75 of Lewis and Langford. The author proposes to replace Vredenduin's axiom $V18$ by the stronger axiom $\sim\Diamond(p\sim q)\cdot\Diamond p : \prec \cdot \Diamond q$; he shows that in the system V_2 obtained thereby only paradoxical propositions of Lewis and Langford are not provable. He also shows that the system E_2 suggested by Emch [J. Symbolic Logic 1, 26–35, 58 (1936); 2, 78–81 (1937)] is equivalent to V_2, if V_2 is completed by the definition $\bigcirc p = \sim(p \prec \sim p)$, while Emch's logical implication and logical equivalence are identified with Vredenduin's $\prec$ and $=$ respectively. Systems V_2 and E_2 are contained in S_2, but they contain the system S_2^0 which results from S_2 when $\Diamond$ is replaced throughout by $\bigcirc$. Some of the results of Parry [J. Symbolic Logic 4, 137–154 (1939); these Rev. 1, 131] are extended to systems resulting from V_2 in the same way as S_4, $S_{4.3}$ and S_5 result from S_2. *A. Heyting.*

Kreisel, G. Note on arithmetic models for consistent formulae of the predicate calculus. Fund. Math. 37, 265–285 (1950).

Die Widerspruchsfreiheit eines Axiomensystems in der Normalform (1) $\bigwedge_{\mathfrak{x}}\bigvee_{\mathfrak{y}}B(\mathfrak{x},\mathfrak{y})$ mit $\mathfrak{x}=x_1,\cdots,x_r$; $\mathfrak{y}=y_1,\cdots,y_s$ ist äquivalent mit der Verifizierbarkeit von (2) $\bigwedge_n q(n)=0$ für eine primitiv-rekursive Funktion q in Z_μ [Hilbert und Bernays, Grundlagen der Mathematik, Band 2, Springer, Berlin, 1939]. Ist (1) widerspruchsfrei, dann gibt es eine Formel B^* in Z_μ, sodass (3) $\bigwedge_n q(n)=0\rightarrow\bigwedge_{\mathfrak{x}}\bigvee_{\mathfrak{y}}B^*(\mathfrak{x},\mathfrak{y})$ in Z_μ beweisbar ist. Verf. benutzt dieses arithmetische Modell von (1) um—unter Voraussetzung der Darstellbarkeit von Z_μ und der ω-Widerspruchsfreiheit—die Existenz einer unentscheidbaren Formel U zu beweisen. Ob U ω-widerspruchsfrei ist, bleibt dabei, im Gegensatz zum Gödelschen Beweis, offen. Anschliessend diskutiert Verf. das Diagonalverfahren in seinem Beweis und in der Skolemschen Paradoxie. *P. Lorenzen* (Bonn).

Rosser, J. B., and Turquette, A. R. Axiom schemes for m-valued functional calculi of first order. II. Deductive completeness. J. Symbolic Logic 16, 22–34 (1951).

Für den im Teil I [J. Symbolic Logic 13, 177–192 (1948); diese Rev. 10, 420] definierten m-wertigen Prädikatenkalkül wird die deduktive Vollständigkeit definiert durch die Beweisbarkeit jeder "analytischen" Formel, d.h. jeder Formel, die für alle Einsetzungen von Wahrheitswerten und Wahrheitswertfunktionen einen ausgezeichneten Wert annimmt. Der Beweis der deduktiven Vollständigkeit wird nach der Methode von Henkin [ibid. 14, 159–166 (1949); diese Rev. 11, 487] geführt. *P. Lorenzen* (Bonn).

Rose, Alan. Conditioned disjunction as a primitive connective for the m-valued propositional calculus. Math. Ann. 123, 76–78 (1951).

Post [The Two-Valued Iterative Systems of Mathematical Logic, Princeton University Press, 1941; these Rev. 2, 337] and Church [Portugaliae Math. 7, 87–90 (1948); these Rev. 10, 421] have shown that the ternary conditioned disjunction function $XY+Y'Z$, together with the logical constants t and f, form a complete set of independent connectives for the two-valued propositional calculus, and that the dual of any formula expressed in terms of these connectives is obtained by interchanging X with Z and t with f. The author generalizes this to the m-valued calculus. He considers the generalized conditioned disjunction function $[Y, X_1, X_2, \cdots, X_m, Y]$ which has the same truth value as X_n when Y has the truth value n. He shows that this function and the truth values 1 to m form a complete set of independent connectives for the m-valued calculus. He defines the notion of dual for this calculus, and shows that the dual of a formula expressed in terms of his primitive connectives is obtained by reversing the order of the letters X_1 to X_m and of the truth values 1 to m. *O. Frink.*

Rose, Alan. A new proof of a theorem of Dienes. Norsk Mat. Tidsskr. 33, 27–29 (1951).

The theorem of Dienes referred to [J. Symbolic Logic 14, 95–97 (1949); these Rev. 11, 1] states that if an identically true formula of the 2-valued propositional calculus expressed in terms of implication and negation is interpreted as a formula of the m-valued calculus, its truth value is always greater than or equal to $\frac{1}{2}$ if m is odd, and greater than $\frac{1}{2}$ if m is even. The author proves this by means of an interpretation of the m-valued calculus in which the elements are sets of $m-1$ elements of a Boolean algebra. This is a generalization of a similar characterization of the three-valued calculus in terms of pairs of elements of a Boolean algebra given by the author in a previous paper [J. London Math. Soc. 25, 255–259 (1950); these Rev. 12, 663].
O. Frink (State College, Pa.).

Zubieta R., Gonzalo. On the substitution of functional variables in the functional calculus of the first order. Bol. Soc. Mat. Mexicana 7, 1–21 (1950). (Spanish)

It is well known that the formulation, in the predicate calculus of first order, of the rule for substituting a formula $\mathfrak{B}$, considered as a function of $x_1, \cdots, x_n$, for an n-adic predicate variable f in a formula $\mathfrak{A}$ entails great difficulties because variables can collide in rather unexpected ways. In fact, practically all the formulations published before 1940, even by the most competent authors, were incorrect. By the use of the idea of functional abstraction, however, a correct formulation can easily be given; and this is undoubtedly the basis of the correct formulation given by Church [Introduction to Mathematical Logic. I., Princeton University Press, 1944, pp. 57, 63; these Rev. 6, 29]. It then follows, as a corollary of Church's rules for the abstraction operator, that two sorts of collisions are to be avoided, viz.: (1) between the arguments of f in $\mathfrak{A}$ and the bound variables of $\mathfrak{B}$, and (2) between the free variables of $\mathfrak{B}$, other than $x_1, \cdots, x_n$, and the bound variables of $\mathfrak{A}$. In the present article the author gives a formulation independent of functional abstraction. This is based on a careful formulation of the syntax of the system in terms of the concatenation operation and recursive definitions. By a rather complex series of such definitions he obtains what amounts to a redefinition of the operation of changing free variables in such a way that collisions of the first sort are automatically avoided. The second sort remains and has to be excluded by hypothesis in his principal theorem. He also shows that his definition is semantically satisfactory in the sense that if a formula is valid for an interpretation (which is defined technically in a partially formalized manner), then any special case of it (in the author's technical sense) is valid also.
H. B. Curry (State College, Pa.).

Fadini, Angelo. Un particolare calcolo funzionale: l'algebra della logica. Ricerca, Napoli 1, no. 4, 34–38 (1950).

Janiczak, Antoni. A remark concerning decidability of complete theories. J. Symbolic Logic 15, 277–279 (1950).

Let L be (1) a consistent formal language, and (2) be complete in the sense that for each sentence of L either it or its negative is provable. Further, (3) let the set of Gödel numbers of the sentences of L be general recursive, (4) let the set of Gödel numbers of the axioms of L be recursively enumerable, (5) let (roughly stated) the set of the rules of inference be general recursive, and (6) let there be a general recursive function Neg such that, if x is the Gödel number of a sentence of L, Neg (x) is the Gödel number of its negation. A language L is said to be decidable if the set of Gödel numbers of its provable sentences is general recursive. The author proves that any L satisfying (1)–(6) is decidable.
R. M. Martin (Philadelphia, Pa.).

Kalmár, László. Eine einfache Konstruktion unentscheidbarer Sätze in formalen Systemen. Methodos 2, 220–231 (1950). (German and English)

The first incompleteness theorem of Gödel is derived very simply for a formal system S subject to three general hypotheses which are essentially as follows: (1) The natural

numbers can be formulated in S, and also, for each natural number n, the properties $x=n$ and $x\neq n$, where x is one of a certain class of terms called numerals, in such a way that if $x=n$ is provable in S, then $x\neq m$ is also derivable for every $m\neq n$. (2) For each term f of a certain class of terms, called function terms, and for each n, there is formulated in S a numeral $f(n)$ and a sentence $f\neq n$, such that if $f\neq n$ is provable, then $f(m)\neq n$ is provable for every m. (3) The function terms and the proofs can be enumerated within S; and there is a function term g, such that if n is the number of a proof of $f\neq m$, where m is also the number of f, then $g(n)=m$ is provable, otherwise $g(n)=0$ is provable. The author then shows that, if q is the number of g, $g(n)\neq q$ is provable for every n; but if $g\neq q$ is provable, then the system is inconsistent. The author indicates that the second part of his third hypothesis can be replaced by a simpler condition involving recursive functions (or even a subclass of them which he calls "elementary functions") which can be verified by an arithmetization process; and he points out that his conditions can be interpreted quite broadly.

H. B. Curry (State College, Pa.).

Robinson, Raphael M. Arithmetical definitions in the ring of integers. Proc. Amer. Math. Soc. **2**, 279–284 (1951).

Im Ring der ganzen Zahlen kann die Menge N der nichtnegativen Zahlen "arithmetisch" definiert werden, z.B. mit zwei Quantoren durch

$$a\varepsilon N \leftrightarrow \mathsf{V}_x \mathsf{V}_y (x^2=a \vee (y^2=1+ax^2 \wedge y\neq 0 \wedge y\neq 1)).$$

Verf. beweist mit Hilfe eines Skolemschen Satzes über die Dichte ganzzahliger Lösungen analytischer Gleichungen, dass N nicht mit nur einem Quantor arithmetisch definierbar ist.

P. Lorenzen (Bonn).

Robinson, Raphael M. Undecidable rings. Trans. Amer. Math. Soc. **70**, 137–159 (1951).

Unter Benutzung des Satzes, dass die Arithmetik unentscheidbar ist, d.h., dass kein Verfahren existiert, das über die Ableitbarkeit jeder Aussage aus den Axiomen der Arithmetik entscheidet, wird das Entscheidungsproblem für Integritätsbereiche I behandelt. Ist in der Theorie über I der Begriff natürliche Zahl definierbar, dann ist auch I unentscheidbar. In dem Polynomring über einem Körper der Charakteristik 0 lassen sich z.B. die natürlichen Zahlen als gewisse Konstanten kennzeichnen. In quadratischen Zahlkörpern werden Sachverhalte wie die Darstellbarkeit jeder total-positiven Zahl als Summe von vier Quadraten herangezogen. In I mit Primelementen p (z.B., 5 im Ring der ganzen Zahlen) können die Potenzen $1, p, p^2, \cdots$ ein Modell der Arithemetik bilden. Addition und Multiplikation der Arithmetik können durch Addition und Teilbarkeit definiert werden und diese sind im Modell definierbar wegen $p^{m+n}=p^m\cdot p^n$ und $m|n\leftrightarrow(p^m-1)|(p^n-1)$. Für algebraische Zahlkörper mit nur einer Fundamentaleinheit θ können ebenso die Potenzen $1, \theta, \theta^2, \cdots$ ein Modell bilden. Unter Benutzung eines "wesentlich unentscheidbaren" Satzes von Mostowski und Tarski ergibt sich die Unentscheidbarkeit jedes Polynomringes über einem I. Aus der Unentscheidbarkeit eines Ringes I folgt in den meisten Fällen die Unentscheidbarkeit des Ringes $I\{x\}$ aller (formalen) Potenzreihen über I, weil I in $I\{x\}$ definiert werden kann. Zum Schluss beweist Verf. dass z.B. bei Polynomringen in mehreren Variablen über endlichen Konstantenkörper kein Modell für die Arithmetik eindeutig ausgezeichnet werden kann.

P. Lorenzen (Bonn).

Novak, I. L. A construction for models of consistent systems. Fund. Math. **37**, 87–110 (1950).

Let (S) be a formal system containing one range of variables $x, y, \cdots$ and a finite number of connectives, and comprising the ordinary predicate calculus. It is supposed that the syntax (Σ) of (S) is expressible in elementary arithmetic; moreover, that (Σ) contains the hypothesis "S is consistent" and that (Σ) is consistent. (S_ϵ) is obtained from (S) by adjoining the Hilbert ϵ-operator. (S') is an extension of (S) which bears to (S) a relation analogous to that existing between the Zermelo-Fraenkel set-theory and the von Neumann-Bernays set-theory as discussed by Hao Wang [Proc. Nat. Acad. Sci. U. S. A. **35**, 150–155 (1949); these Rev. **10**, 670]; thus (S') contains an additional kind of variables $X, Y, \cdots$ and a new connective o such that $X\text{o}y$ is a well formed function. Let $\phi_1, \phi_2, \cdots$ be the statements of (S_ϵ) arranged in some order; a truth-predicate T is defined by (i) and (ii) as follows: (i) $T\phi_1 \equiv \sim Th_{S_\epsilon}(\sim\phi_1)$; (ii) $T\phi_n \equiv \sim Th_{S_\epsilon}(\phi_1\cdots\phi_k\cdot\supset\cdot\sim\phi_n)$, where $\phi_1, \cdots, \phi_k$ are all the statements preceding ϕ_n for which $T\phi_i$ $(1\leqq i\leqq k)$. Here $\ulcorner Th_{S_\epsilon}\zeta\urcorner$ is defined as $\ulcorner\zeta$ is a theorem of $(S_\epsilon)\urcorner$. It is shown that T is definable in the syntax of (S_ϵ). Now let $\phi_\epsilon, \psi_\epsilon, X_\epsilon$ be variables in (Σ), ranging over ϵ-terms. By means of the theorem $\ulcorner Th_S(\exists\alpha)\phi\supset(\exists X_\epsilon)(Th_{S_\epsilon}S_\alpha{}^{X_\epsilon}\phi)\urcorner$ and the analogous theorem for the all-operator, any statement in (S_ϵ) is transformed into a statement in (Σ). The next step is to replace everywhere "Th_{S_ϵ}" by "T." The statements $\ulcorner\phi_M\urcorner$ which are so obtained from theorems of (S) form a model (S_M) of (S). The model (S_M) is enlarged to a model (S_M') of (S') by adjoining to it the transform of theorem-schemata of (S) which can be written in the form

$$\ulcorner(\phi_1)(\cdots)(\exists\phi_k)Th_S((\alpha_1)(\cdots)(\exists\alpha_n)\psi)\urcorner.$$

As (S_M') is denumerable, this gives also a new proof for the Löwenheim-Skolem theorem.

A. Heyting.

Mostowski, Andrzej. Some impredicative definitions in the axiomatic set-theory. Fund. Math. **37**, 111–124 (1950).

If (S) is the Zermelo-Fraenkel set theory and (S') the Bernays-Gödel set theory, then, according to results of Hao Wang and of Novak [see the preceding review and reference cited there], if (S) is consistent, then (S') is consistent. Since Novak's proof is formalizable in (S'), the consistency of (S) cannot be proved in (S'). On the other hand, as is shown here, the definition of truth for (S) is formalizable in (S') and an expression $V(x_1)$ of (S') can be found such that $\ulcorner V(n)\urcorner$ is provable in (S') if n is the Gödel number of a theorem of (S). However, the theorem $\ulcorner(x_1)[x_1$ is the Gödel number of a theorem of $(S)\supset V(x_1)]\urcorner$, though expressible in (S'), is not provable in (S') provided (S') is consistent. If "$\theta(x_1)$" denotes the formula corresponding to $\ulcorner x_1$ is an integer and every theorem of the x_1th order is true$\urcorner$, then $\ulcorner\theta(1)\urcorner$ and $\ulcorner(n)[\theta(n)\supset\theta(n+1)]\urcorner$ are provable, but not $\ulcorner(n)\theta(n)\urcorner$. Finally, if $\phi(x)$ corresponds to $\ulcorner x$ is an integer and $\sim\theta(x)\urcorner$, then $\ulcorner(\exists X)(x)[x\varepsilon X\equiv\phi(x)]\urcorner$ is not provable in (S').

A. Heyting (Amsterdam).

Myhill, John. Report on some investigations concerning the consistency of the axiom of reducibility. J. Symbolic Logic **16**, 35–42 (1951).

This paper is concerned with ideas due to Chwistek and his students. A form of ramified type theory is developed and proved consistent via Fitch's method [J. Symbolic Logic **13**, 95–106 (1948); these Rev. **9**, 559]. A form of

reducibility axiom is provable in the system, and a form of an axiom of choice is provable in a slightly enlarged system. But only a restricted theory of real numbers is forthcoming in either system. No attempt is made here to determine the power of the metalinguistic tools being employed. Hence there is no way of evaluating the convincingness of this consistency "proof". It is still controversial whether Chwistek's ideas are clearly formulable without violating the distinction between mention and use. The present paper does not appear to settle the matter. *R. M. Martin.*

Myhill, John. Towards a consistent set-theory. J. Symbolic Logic **16**, 130–136 (1951).

This paper is a continuation of that sketched in the preceding review. The author is concerned here with a form of axiomatic set theory. After some preliminary simplifications of notation and some definitions, analogues of some of the axioms of Bourbaki [same J. **14**, 1–8 (1949); these Rev. **11**, 73] are derived within S'. But, as in the system of Fitch [ibid. **13**, 95–106 (1948); **14**, 9–15 (1949); these Rev. **9**, 559; **10**, 669], no analogue of the extensionality axiom is present.

R. M. Martin (Philadelphia, Pa.).

Hasenjaeger, G. Ein Beitrag zur Ordnungstheorie. Arch. Math. Logik Grundlagenforsch. **1**, 30–31 (1950).

This paper is concerned with the Hilbert and Bernays axiom-system (B) [See Grundlagen der Mathematik, vol. 1, Springer, Berlin, 1934, p. 273]. The author proves that $B5$, the law that if a is less than b then either the successor of a equals b or the successor of a is less than b, is derivable from the other axioms, provided we take "$a=a$" as an axiom in place of the original "$0=0$". The proof is by elementary induction. *R. M. Martin* (Philadelphia, Pa.).

Zich, O. V. Sur la notion du nombre entier. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. **49** (1948), 139–149 (1950).

In his famous Tractatus Logico-Philosophicus [K. Paul, Trench, Trubner & Co., London, 1922], Wittgenstein raised some objections to the identity or equality concept on philosophical and epistemological grounds. His suggestions were somewhat vague and have never been rigorously formulated within a formal system. The present paper is concerned with formalizing a portion of elementary number theory without the identity sign in accord with Wittgenstein's suggestions. The attempted construction is, however, unsuccessful. The author's leading idea is acknowledged to be due to Kolmogoroff (Rule VI, p. 142). Rule VI, however, is stated very vaguely, and at best seems to be a mere heuristic principle. It is semantical in character, being concerned with designation, and hence is hardly suitable to play the role of a syntactical rule of formation which it is intended to do. The author's axioms for the propositional calculus are those of Hilbert and Ackermann. His axioms for quantifiers seem to be defective. One important quantificational axiom seems to be omitted and no mention is made of a rule of generalization. Furthermore it is by no means clear from the one axiom of quantification given precisely how the Wittgenstein-Kolmogoroff suggestion is supposed to work. Also Rule VII, an intended rule of substitution, needs considerable emendation. The author next sketches what is essentially the classical, Principia Mathematica manner of building up cardinal number theory and then proceeds to his own reconstruction. The definition given, intended to be the classical one of the cardinal number two (p. 144), is defective, a crucial distinctness clause being omitted.

As a sample of an identity sentence in elementary number theory, the author considers "$a+b=c$" and tries to show how the effect of this can be achieved in his identity-free system. Where "a", "b", etc., are constants, possibly his method can be made workable. It is not clear however that these are intended to be constants. Furthermore, the crucial definitions (bottom of p. 145) seem to contain a free variable in the definienda not free in the definientia in a way that is suspicious. It is a consequence, e.g., of these definitions that for any cardinal numbers a, b, and for any properties or classes F, G, F has a members if and only if G has b members. The author allows quantifiers over entities of second level. Hence the Leibnitz definition of identity can be given in his system, and the purpose of the system collapses. In conclusion there are some comments on one-to-one correspondence.

R. M. Martin (Philadelphia, Pa.).

Destouches, J. L. Sur la mécanique classique et l'intuitionnisme. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 74–79 (1951).

Während in der klassischen Physik die Aussage, dass sich ein System S zur Zeit t_0 in einem Punkt P_0 des Phasenraums R befindet: $P(S, t_0)=P_0$, sinnvoll ist, werden in der Quantenmechanik [G. Birkhoff und J. von Neumann, Ann. of Math. (2) **37**, 823–843 (1936)] nur Aussagen der Form $P(S, t_0)\epsilon E$ für messbare Mengen E (mit positiven Mass) aus R zugelassen. Wird als "Negation" dieser Aussage nicht $P(S, t_0)\epsilon -E$, sondern $P(S, t_0)\epsilon -\bar{E}$ (abgeschlossene Hülle) genommen, dann bilden die Aussagen eine Brouwersche Logik, d.h. einen relativ-orthokomplementären Verband. Verf. entnimmt daraus, dass die intuitionistische Mathematik für die Physik adäquat ist. *P. Lorenzen.*

Destouches-Février, P. Sur l'intuitionnisme et la conception strictement constructive. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 80–86 (1951).

Die positive Logik (ohne iterierte Implikationen) wird gedeutet mit Hilfe "zu realisierender Konstruktionen" statt "Aussagen". Die Negation wird nicht eingeführt, weil dies, nach Ansicht der Verf., nicht konstruktiv geschehen könne.

P. Lorenzen (Bonn).

ALGEBRA

Riordan, John. Triangular permutation numbers. Proc. Amer. Math. Soc. **2**, 429–432 (1951).

L'auteur présente diverses interprétations combinatoires des nombres rencontrés par Euler [Opera Omnia, Ser. I, vol. X, p. 373, Teubner, Leipzig-Berlin, 1913] sous la définition: $A(n, p)=\sum_{t=0}^{p-1}(-)^t\binom{n}{t}(p-t)^{n-1}$. Désignant $A(n+1, x)$ par $r(x, n)$, il forme, par voie combinatoire, la fonction génératrice $R_n(t)$ de r et montre qu'elle vérifie l'équation (symbolique) aux différences: $R_n=(t-1)R_{n-1}+(R+R)^{n-1}$. L'intégration est obtenue en posant: $W=\sum_{n=0}^{\infty}R_nu^n/n!$, et donne: $R_n=(1-t)^{n+1}\sum_{k=0}^{\infty}t^kk^n$. De là, on tirerait, à vue, le coefficient cherché r de t^x. Mais l'auteur préfère exprimer R, puis r, en fonction des nombres de Stirling de 2ᵉ espèce, ce qui lui permet de trouver l'expression:

$$r=\sum_{j=1}^{x}(-)^{x-j}\binom{n-j}{x-j}j!S(j,n),$$

plus condensée que la formule de Toscano; puis, par un nouveau développement, il obtient r sous la forme A. La relation de récurrence entre les r est amenée par différentiation de R_n. *A. Sade* (Marseille).

Fadini, Angelo. Il triangolo di Tartaglia nel corpo C [2]. Ricerca, Napoli 1, no. 1, 23–28 (1950).

Glaser, Vladimir. About some relations between determinants. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 5, 162–165 (1950). (Serbo-Croatian. English summary)

Price, G. Baley. Bounds for determinants with dominant principal diagonal. Proc. Amer. Math. Soc. 2, 497–502 (1951).

A determinant $|a_{ij}|$ is said to have dominant principal diagonal if the absolute value of each term on the principal diagonal is greater than the sum of the absolute values of all other terms in the same row. It is known that such determinants do not vanish. Here upper and lower bounds are given for such determinants, the lower bound being an improvement on a result of Ostrowski [Bull. Sci. Math. (2) 61, 19–32 (1937)]. The theorems can also be applied to more general determinants, namely, those for which a set of constants r_k exists, $k=1, \cdots, n-1$, such that the determinants obtained from (a_{ij}), $i, j=k, \cdots, n$, by dividing the first column by r_k, has a dominant principal diagonal.

O. Todd-Taussky (Washington, D. C.).

Afriat, S. N. Bounds for the characteristic values of matrix functions. Quart. J. Math., Oxford Ser. (2) 2, 81–84 (1951).

For any square complex matrix $\mathbf{a}$ the author introduces the "upper modulus" $|\mathbf{a}|^*$ and "lower modulus" $|\mathbf{a}|_*$, whose squares are the maximum and minimum characteristic values of the Hermitian matrix $\bar{\mathbf{a}}'\mathbf{a}$. He proves some basic inequalities about these, and obtains a general theorem: "For a matrix function $\mathbf{f}=\mathbf{f}(\mathbf{a}_1, \mathbf{a}_2, \cdots)$ formed from sums and products in a set $\mathbf{a}_1, \mathbf{a}_2, \cdots$ of matrix arguments, if $f^*=f^*(\mathbf{a}_1, \mathbf{a}_2, \cdots)$ is the corresponding scalar function in the upper moduli $|\mathbf{a}_1|^*, |\mathbf{a}_2|^*, \cdots$ of the arguments, then $\mathbf{f}$ is defined whenever f^* is, and, if ϕ is any characteristic value of $\mathbf{f}$, then $0\leqq|\phi|=f^*$." The author's $|\mathbf{a}|^*$, $|\mathbf{a}|_*$ coincide with the upper and lower bounds of von Neumann and Goldstine [Bull. Amer. Math. Soc. 53, 1021–1099 (1947); these Rev. 9, 471], who prove the basic inequalities. Afriat states without special discussion that his results apply also to matrices of infinite order. *G. E. Forsythe.*

Drazin, M. P., Dungey, J. W., and Gruenberg, K. W. Some theorems on commutative matrices. J. London Math. Soc. 26, 221–228 (1951).

The authors consider square matrices with elements in the complex field, and begin with an elementary proof of the following result of the reviewer [Bull. Amer. Math. Soc. 42, 592–600 (1936)]. If $A_1, \cdots, A_m$ is a given set of $n\times n$ matrices, then the following statements are equivalent: (i) for every scalar polynomial $p(x_1, \cdots, x_m)$ in the (non-commutative) variables $x_1, \cdots, x_m$, each of the matrices $p(A_1, \cdots, A_m)(A_iA_j-A_jA_i)$, $i, j=1, 2, \cdots, m$, is nilpotent; (ii) there is a (unitary) matrix P such that each matrix $P^{-1}A_iP$ is triangular; (iii) there is an ordering of the eigenvalues $\alpha_k^{(i)}$ of each A_i such that the eigenvalues of any (scalar) rational function $R(A_1, \cdots, A_m)$ of $A_1, \cdots, A_m$ are $R(\alpha_k^{(1)}, \cdots, \alpha_k^{(m)})$, $k=1, 2, \cdots, n$. Their proof is based primarily on a lemma, which is proved by induction, which states that if condition (i) is satisfied, then $A_1, \cdots, A_m$ have a common eigenvector. It is also shown that if each of the matrices $A_1, \cdots, A_m$ commutes with each matrix $A_iA_j-A_jA_i$, $i=1, 2, \cdots, n$, then condition (i) holds. This is also essentially due to the reviewer [Trans. Amer. Math. Soc. 36, 327–340 (1934)]. Moreover, under these commutation conditions, given any eigenvalue α of A_1, there exists a common eigenvector of $A_1, \cdots, A_m$ having eigenvalue α for A. Several other related results are also obtained.

N. H. McCoy (Northampton, Mass.).

Seebach, Karl. Über ein vollständiges System von Bewegungsinvarianten der Hyperflächen zweiter Ordnung im $\mathfrak{R}_n$. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1950, 143–155 (1951).

Let $\mathfrak{A}$ be a real symmetric matrix of order $n+1$ associated with an inhomogeneous quadratic form and $\mathfrak{A}_1$ its principal minor of order n giving the homogeneous part. Then $R=\text{rank }\mathfrak{A}$, $r=\text{rank }\mathfrak{A}_1$, the nonzero eigenvalues of $\mathfrak{A}_1$, and $A_\rho=$ the coefficient of $\mu^{n+1-\rho}$ in $|\mathfrak{A}-\mu I|$ for $\rho=\max(R, r+1)$ are a complete system of invariants under rotations and translations. *W. Givens* (Knoxville, Tenn.).

Rutherford, D. E. Compound matrices. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 16–22 (1951).

Let B be a matrix of order $\binom{n}{k}\times\binom{m}{k}$ and of rank $\sigma=0, 1$ or $\binom{\rho}{k}$ for $\rho>k$. If it is known that B is the kth compound $A^{(k)}$ of an $n\times m$ matrix A, the components of A are found (to within a common kth root of unity when $\sigma>1$) in terms of those of B. Conditions that $B=A^{(k)}$ for some A are given: for rank $B=1$, the rows and columns of B must satisfy the quadratic relations characterizing the (Grassmann) coordinates of a linear space; for rank $B>1$, only part of the complicated conditions are given explicitly. *W. Givens.*

Kotelyanskiĭ, D. M. On the theory of nonnegative and oscillating matrices. Ukrain. Mat. Žurnal 2, no. 2, 94–101 (1950). (Russian)

If no minor of a square matrix $(a_{ij})_1^n$ is negative, then for every p, $0<p<n$ [F. R. Gantmacher and M. G. Kreĭn, Oscillation matrices and small oscillations of mechanical systems, GONTI, Moscow-Leningrad, 1941; these Rev. 3, 242], $|a_{ij}|_1^n\leqq|a_{ij}|_1^p\cdot|a_{ij}|_{p+1}^n$. The author proves that the same inequality holds if merely (1) no a_{ij} is negative, (2) no principal minor is negative, (3) every minor obtained by augmenting (bordering) a principal minor with one more row index and one more column index is nonnegative. The author's second result gives a structure theorem for non-singular matrices, no minor of which is negative. For symmetric matrices of this last type, the structure must be diag $(A_1, A_2, \cdots, A_m)$, where no power A_i^s has a negative or zero minor. *J. L. Brenner* (Pullman, Wash.).

Foulkes, H. O. Modified bialternants and symmetric function identities. J. London Math. Soc. 25, 268–275 (1950).

Hirsch has shown [J. London Math. Soc. 24, 144–145 (1949); these Rev. 10, 671] that, if every power α_i^k in the bialternant

$$a_{n-k}=|\alpha_i^n, \alpha_i^{n-1}, \cdots, \alpha_i^{k+1}, \alpha_i^{k-1}, \cdots, \alpha_i, 1|/\prod_{i<j}(\alpha_i-\alpha_j)$$

is replaced by $\alpha_i^{(k)}=(\alpha_i+\beta_1)(\alpha_i+\beta_2)\cdots(\alpha_i+\beta_k)$, the result

reduces to $a_m' = a_m H_0 + a_{m-1} H_1 + \cdots + a_1 H_{m-1} + a_0 H_m$, where $m = n-k$ and H_r is the homogeneous product sum of degree r of $\beta_1, \beta_2, \cdots, \beta_{k+1}$. A negative sign is attached to each β_i in the present paper. The author gives an alternative proof of Hirsch's result and evaluates the modified bialternant h_m' obtained from

$$h_m = |\alpha_i^{m+n-1}, \alpha_i^{n-2}, \alpha_i^{n-3}, \cdots, \alpha_i, 1| / \prod_{i<j} (\alpha_i - \alpha_j)$$

by replacing α_i^k by $\alpha_i^{(k)}$. Finally, the author evaluates the modified bialternant obtained from any S-function $\{\lambda\}$ by replacing α_i^k by $\alpha_i^{(k)}$. The proofs employ a theorem on determinants due to E. Beltrami and an extension given by G. Garbieri [see Muir, The Theory of Determinants . . . , vol. 3, Macmillan, London, 1920, pp. 153, 163–165].

G. B. Price (Lawrence, Kans.).

Foulkes, H. O. Reduced determinantal forms for S-functions. Quart. J. Math., Oxford Ser. (2) **2**, 67–73 (1951).

It is well known that S-functions or bi-alternants can be expressed as determinants whose elements are the homogeneous product sums h_r, or the elementary symmetric functions a_r. Thus

$$\{\lambda_1, \cdots, \lambda_n\} = |h_{\lambda_s - s + t}| = |a_{\mu_s - s + t}|$$

where $(\mu_1, \cdots, \mu_m)$ is the partition conjugate to $(\lambda_1, \cdots, \lambda_n)$. The reviewer also obtained determinantal forms of lower degree of which the elements are S-functions of rank 1, i.e., of the form $\{r, 1^s\}$. The author gives a simple derivation of these reduced forms. Thus, since

$$\begin{aligned}\{p, 1^a\} &= h_p a_a - h_{p+1} a_{a-1} + \cdots \pm h_{p+a} \\ &= h_{p-1} a_{a+1} - h_{p-2} a_{a+2} + \cdots \pm a_{p+a},\end{aligned}$$

then e.g.,

$$\{432^21\} = \begin{vmatrix} a_5 & a_6 & a_7 & a_8 \\ a_3 & a_4 & a_5 & a_6 \\ 1 & a_1 & a_2 & a_3 \\ \cdot & \cdot & 1 & a_1 \end{vmatrix} = \begin{vmatrix} a_5 & a_6 & a_7 & a_8 \\ a_3 & a_4 & a_5 & a_6 \\ 1 & a_1 & a_2 & a_3 \\ \cdot & \cdot & 1 & a_1 \end{vmatrix} \cdot \begin{vmatrix} -h_3 & h_2 & -h_1 & 1 \\ h_2 & -h_1 & 1 & \cdot \\ -h_1 & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{vmatrix}$$

$$= \begin{vmatrix} -\{41^4\}, & \{31^4\}, & -\{21^4\}, & \{1^6\} \\ -\{41^2\}, & \{31^2\}, & -\{21^2\}, & \{1^3\} \\ \cdot & \cdot & \cdot & 1 \\ \cdot & 1 & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} \{41^4\}, & \{21^4\} \\ \{41^2\}, & \{21^2\} \end{vmatrix}.$$

The same result may be obtained from the determinant $|h_{\lambda_s - s + t}|$. Frobenius' nomenclature for a partition is very convenient. Thus

$$\{5421\} = \begin{Bmatrix} 42 \\ 31 \end{Bmatrix} = \begin{vmatrix} \begin{Bmatrix} 4 \\ 3 \end{Bmatrix} & \begin{Bmatrix} 2 \\ 3 \end{Bmatrix} \\ \begin{Bmatrix} 4 \\ 1 \end{Bmatrix} & \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} \end{vmatrix} = \begin{vmatrix} \{51^3\} & \{31^3\} \\ \{51\} & \{31\} \end{vmatrix}.$$

A similar derivation is given for certain hybrid forms given by the reviewer for which some columns of the determinant contain S-functions of rank 1, while other columns contain either a_r's or h_r's. [See D. E. Littlewood, Group Characters and Matrix Representations of Groups, Oxford University Press, 1940, pp. 112, 114; these Rev. **2**, 3.] All minors of the reduced form determinant are simple S-functions. Hence the various compound matrices yield determinants with S-function elements which represent powers of the original determinant. Application is made to differential operators associated with S-functions [Foulkes, J. London Math. Soc. **24**, 136–143 (1949); these Rev. **11**, 4]. A bordered determinantal form is given for $D_{m1^n}\{\lambda\}$.

D. E. Littlewood (Bangor).

✱Okunev, L. Ya. Problema resol'vent Čebotarëva. [Čebotarev's Problem of Resolvents]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949. 55 pp. (1 plate).

This brochure, the first of a series designed to popularize the achievements of Soviet scientists in mathematics and physics, is a descriptive account, preceded by a short biography and introduction to the concepts of group, field, solvability by radicals, and Galois group of an equation, of the work of N. G. Čebotarev establishing a lower bound on the number of parameters of a resolvent of a polynomial equation with coefficients which are polynomials over the complex numbers in a number of parameters [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **7**, 123–146 (1943); these Rev. **6**, 113; Jubilee Symposium . . . , v. 1, Moscow-Leningrad, 1947; these Rev. **9**, 638; for a translation into German see Sowjetwissenschaft **1948**, no. 4, 140–152 (1948); these Rev. **12**, 77].

E. R. Kolchin.

Abstract Algebra

✱Jacobson, Nathan. Lectures in Abstract Algebra. Vol. I. Basic Concepts. D. Van Nostrand Co., Inc., Toronto, New York, London, 1951. xii+217 pp. $5.00.

The author, in his preface, indicates that this book is intended as the introductory volume of a general treatise on abstract algebra, later volumes dealing with vector spaces and with the theory of fields, including valuation theory and Galois theory, being projected. To paraphrase the preface again, the present volume is intended to introduce the reader to the main branches of modern algebra, the treatment of each branch being carried sufficiently far to give the reader an understanding of the ideas underlying each branch. In the opinion of the reviewer, the author has been brilliantly successful in his aim, and the completed work should prove a worthy successor to such earlier works as van der Waerden's "Moderne Algebra", to which, allowing for the passage of time since its publication, the present work is similar in scope. An idea of the ground covered by the present volume is conveyed by the headings of the seven chapters. These, following an introduction to the notations and concepts of set theory and the theory of relations used in abstract algebra, are as follows: (1) Semi-groups and groups, (2) Rings, integral domains, and fields, (3) Extensions of rings and fields, (4) Elementary factorisation theory, (5) Groups with operators, (6) Modules and ideals, (7) Lattices.

The first three chapters require little comment. The first chapter deals with elementary group theory up to and including the concept of homomorphism, and presents no unusual features. The second chapter fulfills a similar function for the concepts enumerated in its heading, closing with an account of rings of endomorphisms. The third chapter opens with an account of the extension of a ring in a ring with an identity, and of an integral domain to its quotient field, continues with an account of polynomial rings and the elementary theory of field extensions, both transcendental and algebraic, and concludes with an account of polynomials in several variables, including a treatment of symmetric functions, and rings of functions.

The next chapter is concerned with the ideas clustering around the unique factorisation theorem. A departure from customary practise is that the theory is developed for semi-

groups rather than integral domains, this emphasising the multiplicative nature of the theory. A semigroup is termed Gaussian if the unique factorisation theorem holds in it, and the early part of the chapter is concerned with obtaining conditions, in a form convenient for applications, that a semigroup should be Gaussian. The second half of the chapter is concerned with classes of integral domains which are Gaussian (more accurately, whose multiplicative semigroups of nonzero elements are Gaussian), and it is shown that all principal ideal domains and also all integral domains with a Euclidean algorithm are Gaussian. The chapter concludes with a proof that the property of being Gaussian is preserved under transcendental extension.

The above four chapters comprise what may be considered as the elementary part of the book, the remaining three chapters being on a more advanced level. The first of these, dealing with groups with operators, is essentially a continuation of chapter I, the concepts of that chapter being transferred to groups with operators, and then extended. The theory of homomorphisms is carried on up to the Jordan-Hölder theorem in the form due to Schreier and Zassenhaus. The remainder of the chapter is concerned with the concept of direct product, and contains a detailed account of the ideas underlying the theorem of Krull and Schmidt, which is also proved. The chapter closes with a short account of infinite direct products.

The next chapter opens with an introduction to the theory of modules (i.e., Abelian groups with a ring acting as operator domain), then deals with the chain conditions, as applied to modules and ideals, and this leads naturally to the Hilbert basis theorem for ideals in a ring of polynomials whose coefficients lie in a ring whose left ideals satisfy the ascending chain condition. The second part of the chapter then deals with the ideal theory of Noetherian rings, with proofs of both the uniqueness theorem for the representation of an ideal as the intersection of primary ideals, and of the uniqueness of isolated components. The last section of the chapter is devoted to the study of integral dependence of elements of a ring relative to a subring, a notion generalising the concept of algebraic integer.

The final chapter is a short introduction to the theory of lattices. The main types of lattice are dealt with, including modular lattices, complemented lattices, and Boolean algebras. Certain theorems proved earlier in the book, now appear in their lattice-theoretic form, among them, the Jordan-Hölder theorem. The chapter concludes with an account of the relation between Boolean algebras and a certain type of ring.

The book is well printed, with a pleasant layout, and on the whole, is free from misprints. It also contains numerous examples of varying degrees of difficulty. *D. Rees.*

Hostinsky, L. Aileen. Endomorphisms of lattices. Duke Math. J. 18, 331–342 (1951).

Die invarianten zulässigen Untergruppen einer Operatorgruppe bilden einen vollständigen modularen Verband. Für jede aufsteigende (evtl. transfinite) Folge $b_1, b_2, \cdots$ gilt ausserdem $a \wedge \mathsf{V}_\nu b_\nu = \mathsf{V}_\nu a \wedge b_\nu$. Ein Paar p/p' (mit $p \geqq p'$) eines solchen Verbandes heisst durch η auf q/q' homomorph abgebildet, wenn (1) $(\mathsf{V}_\nu x_\nu)\eta = \mathsf{V}_\nu x_\nu \eta$,

$$(2) \qquad x \varepsilon p/p' \rightarrow (x/p')\eta = x\eta/q',$$

(3) $x_1\eta = x_2\eta \rightarrow$ es gibt x_1', x_2' mit

$$x_1 \vee x_1' = x_2 \vee x_2', \quad x_1'\eta = x_2'\eta = q'.$$

Die Vereinigung der Elemente x mit $x\eta = q'$ bildet den Kern $k(\eta)$. Der Homomorphismus η induziert einen Isomorphismus von $p/k(\eta)$ auf $q/0$. Es gilt der Isomorphiesatz $x_1 \vee x_2/x_1 \simeq x_2/x_1 \wedge x_2$. Für Endomorphismen η $(p\eta \leqq p)$ heisst $k(\eta) \vee k(\eta^2) \vee \cdots$ das Radikal $r(\eta)$. Der Endomorphismus η zerfällt p, wenn $r(\eta)$ direkter Summand ist: $p = s \oplus r(\eta)$. Wenn η alle η-zulässigen Elemente x $(x\eta \leqq x \leqq p)$ zerfällt, ist s eindeutig bestimmt als Vereinigung aller η-automorphen Elemente t ($\eta t = t$ und $x\eta \leqq x \leqq t \rightarrow x\eta = x$). Anschliessend wird der Fittingsche Zerlegungssatz übertragen. *P. Lorenzen* (Bonn).

Thrall, R. M. On the projective structure of a modular lattice. Proc. Amer. Math. Soc. 2, 146–152 (1951).

In einem modularen Verband L mit einer maximalen Kette $\bigcap = \vartheta_0 \subset \vartheta_1 \subset \cdots \subset \vartheta_l = \bigcup$ gibt es genau dann projektive Quotienten $\vartheta_i/\vartheta_{i-1}$ und $\vartheta_j/\vartheta_{j-1}$ wenn L nicht distributiv ist. Unter den Ketten die solche Projektivitäten herstellen, zeichnet Verf. eindeutig die "kanonische" Kette aus. Jeder prime Quotient a/b ist projektiv zu einem Quotienten $\vartheta_i/\vartheta_{i-1}$ und, wenn a/b nicht unmittelbar projektiv zu $\vartheta_i/\vartheta_{i-1}$ ist, gibt es genau 2 Ketten der Länge 2, die die Projektivität herstellt. Es wird auch noch eine andere (nicht eindeutig bestimmte) Normalform für Projektivitäten angegeben. *P. Lorenzen* (Bonn).

Lévy-Bruhl, Jacques. Sur un domaine à trois opérations. C. R. Acad. Sci. Paris 232, 1989–1991 (1951).

The domain D in question is an additive group with two multiplication laws denoted by . and &. It is further assumed that D contains two vector spaces U, V with respect to a field of characteristic 0 with the following product law: $(a_1 \& a_2 \& \cdots \& a_k).(b_1 \& b_2 \& \cdots \& b_k) = \sum a_1 b_{i_1} \& a_2 b_{i_2} \& \cdots \& a_k b_{i_k}$, where all the a_i (b_i) belong to the same one of the two spaces and the sum is over all permutations $i_1, \cdots, i_k$ of $1, \cdots, k$. A given right and left unit which obeys certain rules can be added to D. It can then be proved that any product $P.Q$ is defined where $P = u_1 \& \cdots \& u_{p_1}$, $Q = v_1 \& \cdots \& v_{p_2}$, $u_i \varepsilon U$, $v_j \varepsilon V$ and p_1, p_2 are arbitrary integers. As examples are mentioned (1) the set V of differentiable functions of n variables while U is the set of differential operators, (2) the set of $n \times n$ matrices with certain product definitions, and (3) symmetric functions of n variables. In (3) these concepts give a new approach to the relations found by van der Corput [Math. Centrum Amsterdam, Scriptum no. 3 (1950); these Rev. 11, 74]. *O. Todd-Taussky* (Washington, D. C.).

Arens, Richard. Advanced algebra: operations without numbers. Math. Mag. 24, 253–264 (1951).

An exposition of the definitions and fundamental properties of groups, fields, and rings. *N. H. McCoy.*

Zelinsky, Daniel. Rings with ideal nuclei. Duke Math. J. 18, 431–442 (1951).

Soit R un anneau topologique séparé et complet dans lequel un système fondamental de voisinages de 0 est formé d'une famille d'idéaux bilatères A_α. L'auteur se propose d'étudier la structure de R en le considérant comme limite projective de la famille des anneaux discrets R/A_α; si la structure de ces derniers est connue, et telle que pour $A_\alpha \subset A_\beta$, l'homomorphisme canonique de R/A_α sur R/A_β établisse une "concordance" convenable entre les structures de ces deux anneaux, en un sens naturel, alors on peut "passer à la limite" et obtenir des théorèmes de décomposition pour R. Par exemple, si R est commutatif, et si chacun des anneaux R/A_α satisfait à la condition minimale, alors R est produit (topologique) d'une famille (finie ou infinie)

d'anneaux primaires et d'un anneau identique à son radical (tous les facteurs étant considérés comme discrets); ce dernier facteur disparaît si R a un élément unité, et on peut alors charactériser les facteurs primaires comme des complétions de R pour des topologies moins fines que la topologie initiale, et définies d'une façon analogue aux topologies p-adiques sur les entiers, à partir d'idéaux ouverts, maximaux et réguliers de R. Un autre cas est celui où R est semi-simple (au sens de Jacobson), tel que chacun des R/A_α satisfait à la condition minimale, et où la famille des A_α est dénombrable; alors R est un produit d'une famille d'anneaux simples discrets satisfaisant à la condition minimale (autrement dit, des anneaux de matrices). L'auteur applique ses résultats à l'étude de la topologie d'un anneau de Dedekind quand on prend tous les idéaux comme voisinages, mais omet de remarquer que cette topologie a déjà été déterminée par N. Bourbaki, comme cas particulier d'un résultat plus général [N. Bourbaki, Éléments de mathématique, Topologie générale, chap. III, Hermann, Paris, 1942, p. 60, exerc. 26; ces Rev. 5, 102]. *J. Dieudonné.*

Buck, R. Creighton. Extensions of homomorphisms and regular ideals. J. Indian Math. Soc. (N.S.) 14, 156–158 (1950).

Soit R un anneau, A un idéal de R contenu dans le centre de R; l'auteur remarque que si φ est un homomorphisme de A sur un anneau Ω ayant un élément unité 1, et si $u \in A$ est tel que $\varphi(u)=1$, alors $x \rightarrow \varphi(xu)$ est l'unique homomorphisme de R sur Ω qui prolonge φ. En prenant pour A l'algèbre $L^1(G)$ d'un groupe abélien localement compact G, pour R la sous-algèbre de l'algèbre des mesures bornées sur G, engendrée par A et les mesures ponctuelles, on retrouve la correspondance biunivoque entre caractères continues de A et caractères continus du groupe G. L'auteur montre ensuite que l'application $I \rightarrow I \cap A$ est une correspondance biunivoque entre idéaux réguliers (bilatères) I de R tels que $I+A=R$, et idéaux réguliers de A. *J. Dieudonné.*

***Jaffard, P. Sur les idéaux indépendants.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 95–97. Centre National de la Recherche Scientifique, Paris, 1950.

Ideals $\{I_r\}$ in a ring A are independent if the homomorphism from A into the direct sum of $\{A/I_r\}$ is onto. The author's main result is that if every two ideals are independent, and if each A/I_r equals its square, then any finite number of the ideals are independent. *I. Kaplansky.*

Ballieu, Robert, et Schuind, Marie-Jeanne. Anneaux à module de type (p^m, p^{m+n}). Ann. Soc. Sci. Bruxelles. Sér. I. 65, 33–40 (1951).

Continuing an earlier paper [same Ann. Sér. I. 63, 137–147 (1949); these Rev. 11, 711] the authors investigate rings whose additive group is of type (p^m, p^{m+n}).

I. Kaplansky (Chicago, Ill.).

***Riguet, J. Produit tensoriel de treillis et théorie de Galois généralisée.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 173–178. Centre National de la Recherche Scientifique, Paris, 1950.

Binary relations and tensor products of lattices are discussed along the lines of the author's previous work [C. R. Acad. Sci. Paris 226, 40–41, 143–146 (1948); Bull. Soc. Math. France 76, 114–155 (1948); these Rev. 9, 265; 10, 502]. Here, N-ary relations are generalized to the case of infinitely many variables; they become the single-valued mappings of a set U into a set E. If $\mathfrak{R}$ is a set of such relations then $\mathfrak{A}[\mathfrak{R}]$ denotes the set of permutations of E which are automorphisms of R for every $R \in \mathfrak{R}$. Theorem: Given a group $\mathcal{G}$ and a set U of power not less than the power of E, then $\mathcal{G}=\mathfrak{A}[\mathfrak{A}^{-1}[\mathcal{G}]]$, so that $\mathcal{G}$ is the group of automorphisms of the Boolean algebra $\mathfrak{A}^{-1}[\mathcal{G}]$ of relations invariant under $\mathcal{G}$. $\mathfrak{R}^*=\mathfrak{A}^{-1}[\mathfrak{A}[\mathfrak{R}]]$ provided the power of U is not less than that of E, where $\mathfrak{R}^*$ is the closure of $\mathfrak{R}$ under certain abstract operations which correspond to ordinary tensor operations of addition, scalar multiplication, symmetry, projection, and tensor multiplication.

P. M. Whitman (Silver Spring, Md.).

***Krasner, Marc. Généralisation abstraite de la théorie de Galois.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 163–168. Centre National de la Recherche Scientifique, Paris, 1950.

The author describes without proofs results some of which were published by him earlier [J. Math. Pures Appl. (9) 17, 367–385 (1938)]. Let E be a set and U an auxiliary set at least as big as E. A relation in E is a set of mappings of U into E. A set of relations is logically closed if it is closed with respect to certain operations; every set R of relations has a unique logical closure R_l. The pair $S=(E, R)$ is called a structure on E, and (E, R) is called equivalent to (E, R') if $R_l=R_l'$. The set $G_{E/S}$ of all automorphisms of S (i.e. permutations of E which preserve all relations in R) is called the Galois group of E with respect to S. Every group of permutations of E is a $G_{E/S}$ for a suitable S, and two structures S, S' are equivalent if and only if $G_{E/S}=G_{E/S'}$.

The equivalence class $K=K(S)$ of structures in E which contains S is called the abstract field defined by S. The set $G_{E/S}$ is sometimes denoted by $G_{E/K}$. The set of all $e \in E$ invariant under every element of $G_{E/K}$ is called the domain of rationality of K. If S has certain properties too complicated to describe here, and if K' is the abstract field obtained by adjoining (in a certain sense) a subset $\bar{E}$ of E to $K(S)$, then the domain of rationality of K' is the smallest set in E which contains $\bar{E}$ and the domain of rationality of $K(S)$ and which is closed in a certain sense. Examples of structures S with the required properties are given, including one constructed from classical Galois theory.

E. R. Kolchin (New York, N. Y.).

***Dieudonné, Jean. Progrès et problèmes de la théorie de Galois.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 169–172. Centre National de la Recherche Scientifique, Paris, 1950.

The author observes a parallelism between the essential theorems of classical Galois theory and those of the theory of commutation of simple subalgebras of a simple algebra. He then describes a general Galois theory for simple rings which has as special cases these two parallel theories. The results and detailed proofs have already been published [Comment. Math. Helv. 21, 154–184 (1948); these Rev. 9, 563]. *E. R. Kolchin* (New York, N. Y.).

Krull, Wolfgang. Die Verzweigungsgruppen in der Galoisschen Theorie beliebiger arithmetischer Körper. Math. Ann. 121, 446–466 (1950).

The classical theory of the decomposition, inertial, and ramification groups of a prime ideal in a finite normal ex-

tension of an algebraic number field has been generalized to the case of a field with a valuation by Deuring on the one hand [Math. Ann. **105**, 277–307 (1931)] and by the author himself on the other [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1930**, 225–238; J. Reine Angew. Math. **167**, 160–196 (1932)]. A ramification theory for arbitrary integrally closed local rings without zero divisors was also sketched by the author [S.-B. Phys.-Med. Soz. Erlangen **67/68**, 324–328 (1937)]. In the present paper he gives in detail a theory which covers all these cases, at least as far as the first ramification group is concerned. Higher ramification groups are not defined.

Let R be an integrally closed integral domain, K its quotient field, N a finite normal separable extension of K, G its Galois group, S the integral closure of R in N, $\bar{\mathfrak{p}}$ any maximal ideal in S, $\mathfrak{p}=\bar{\mathfrak{p}}\cap R$. The decomposition group G_z of $\bar{\mathfrak{p}}$ is defined to consist of those automorphisms of N over K which map $\bar{\mathfrak{p}}$ into itself, the decomposition field K_z is the fixed field of G_z, R_z is the integral closure of R in K_z, $\mathfrak{p}_z=\mathfrak{p}\cap R_z$. It is proved that the natural isomorphism of the residue field $\bar{K}=R/\mathfrak{p}$ into the residue field $\bar{K}_z=R_z/\mathfrak{p}_z$ is actually onto.

The elements of G_z induce automorphisms of $\bar{N}=\bar{S}/\bar{\mathfrak{p}}$ over $\bar{K}$. Those inducing the identity constitute the inertial group G_t. With K_t, R_t, $\mathfrak{p}_t$ defined in obvious fashion, the following is proved: The field $\bar{K}_t=R_t/\mathfrak{p}_t$ is the maximal separable extension of $\bar{K}$ in $\bar{N}$; it is normal over $\bar{K}$ and its Galois group is isomorphic to G_z/G_t; $\mathfrak{p}_t=R_t\mathfrak{p}_z$.

Let an ideal A of S be called invariant if it is carried into itself by every element of G_t. Let A and A_1 be invariant ideals such that (1) $A\supset A_1\supseteq\bar{\mathfrak{p}}A$, (2) there is no invariant ideal between A and A_1, (3) A/A_1 is finite-dimensional over $\bar{N}=S/\bar{\mathfrak{p}}$. Then G_t can be represented in natural fashion in the vector space A/A_1, the resulting (obviously irreducible) representation being denoted by $\Delta(A, A_1)$. It is easy to prove the existence of "sufficiently many" such ideal pairs, even with the stronger condition (3′) A is finitely generated (over S); moreover, every representation $\Delta(A, A_1)$ is equivalent to one in which A satisfies (3′). The (first) ramification group G_v is now defined to be the intersection of the kernels of all the representations $\Delta(A, A_1)$. A first justification for this definition is the fact that, as in the classical case, one can prove: If G_v is not the identity, then $\bar{N}$ is of prime characteristic p, and G_v is a p-group. Furthermore, if all the representations $\Delta(A, A_1)$ are 1-dimensional, then G_t/G_v is Abelian and its order is prime to p.

Suppose the ideals in S satisfy the ascending chain condition, and let

$$\bar{\mathfrak{p}}=A_0\supset A_1\supset\cdots\supset A_s=\bar{\mathfrak{p}}^2$$

be a composition series of invariant ideals. Then G_v is equal to the intersection of the kernels of the finitely many representations $\Delta(A_{i-1}, A_i)$. For an algebraic number field this gives $G_v=$kernel of $\Delta(\bar{\mathfrak{p}}, \bar{\mathfrak{p}}^2)$, which is just the classical definition.

Now suppose S is a valuation ring. Then, of course, (a) $\bar{\mathfrak{p}}$ contains all nonunits, and moreover, as is easily proved, (b) every principal ideal is invariant (whence it follows that all the representations $\Delta(A, A_1)$ are 1-dimensional). The author subsequently makes use only of these two properties and not of the valuation itself. He proves that the group G_t/G_v is isomorphic to H_v/H_t, where H_v (H_t) is the group of all fractional principal ideals in N with a generator in K_v (K_t). Specialized to valuation rings, this gives the known results of Krull and Deuring on the relation between the Galois groups and the value groups.

The paper contains various examples and counterexamples. Thus, although G_t/G_v has been proved Abelian in certain important cases, it is shown that it need not be so in general. Indeed, $\bar{N}$ being given, G_t/G_v can be an arbitrary finite group having a completely reducible, faithful representation over $\bar{N}$. A number of unsolved problems are posed, of which we mention only one: Do the representations $\Delta(A, A_1)$ include all irreducible representations of G over $\bar{N}$?

I. S. Cohen (Cambridge, Mass.).

Nakayama, Tadasi. On Krull's conjecture concerning completely integrally closed integrity domains. III. Proc. Japan Acad. **22**, no. 8, 249–250 (1946).

In a previous paper [Proc. Imp. Acad. Tokyo **18**, 185–187 (1942); these Rev. **7**, 236] the author has given an example of a lattice ordered Abelian group which cannot be embedded in a vector group, with preservation of lattice operations. He now shows that there is no possible order embedding.

I. Kaplansky (Chicago, Ill.).

Nakayama, Tadasi. Supplementary remarks on Frobeniusean algebras. I. Proc. Japan Acad. **25**, no. 7, 45–50 (1949).

An algebra A over a field F is called Frobeniusean if it possesses a unit element and if its left and right regular representations are equivalent. In this paper one-sided criteria for Frobeniusean algebras are derived: (1) A is Frobeniusean if and only if a right ideal eA with primitive idempotent e has a unique minimal right subideal; (2) A is Frobeniusean if and only if A possesses a left unit element and R, the left annihilator $l(N)$ of the radical N, is a principal right ideal. An example shows that these results do not hold for rings with minimum condition.

Marshall Hall, Jr. (Washington, D. C.).

Nakayama, Tadasi, and Ikeda, Masatosi. Supplementary remarks on Frobeniusean algebras. II. Osaka Math. J. **2**, 7–12 (1950).

An algebra A over a field F is quasi-Frobeniusean if and only if the dualities

$$(\mathrm{a}_1)\quad l(r(l))=l,\qquad (a_2)\quad r(l(r))=r,$$

between left and right ideals hold, where $l(M)$ and $r(M)$ are respectively the left and right annihilators of the set M. Further, A is Frobeniusean if and only if in addition to (a_1) and (a_2) the rank relations

$$(\mathrm{b}_1)\qquad (l:F)+(r(l):F)=(A:F),$$
$$(\mathrm{b}_2)\qquad (r:F)+(l(r):F)=(A:F)$$

also hold. It is shown that for algebras (but not for rings with minimum condition) condition (a_1) implies (a_2), and (a_1) and (b_1) imply (a_2) and (b_2). Moreover, it is shown that condition (a_1) holds for all ideals in A if it holds for all nilpotent ideals.

Marshall Hall, Jr.

Jacobson, N. General representation theory of Jordan algebras. Trans. Amer. Math. Soc. **70**, 509–530 (1951).

A linear mapping $a\rightarrow S_a$ of a Jordan algebra $\mathfrak{A}$ over a field Φ into the algebra of all linear transformations S of a vector space $\mathfrak{M}$ over Φ is called a representation of $\mathfrak{A}$ if the functional equations

$$S_aS_{bc}-S_{bc}S_a+S_bS_{ca}-S_{ca}S_b+S_cS_{ab}-S_{ab}S_c=0,$$
$$S_aS_bS_c+S_cS_bS_a+S_{(ac)b}=S_aS_{bc}+S_bS_{ca}+S_cS_{ab}$$

are satisfied. The right multiplications of $x\rightarrow xa=xR_a$ satisfy these equations and so $a\rightarrow R_a$ is a representation called the regular representation. The elementary properties

of representations are studied and are connected with Lie algebras and Lie triple systems. The universal associative algebra (of the representations) of any Jordan algebra is defined and is shown to be finite-dimensional when $\mathfrak{A}$ is finite-dimensional. Every representation $S(\mathfrak{A})$ of a Jordan algebra $\mathfrak{A}$ has an enveloping associative algebra $S(\mathfrak{A})^*$. If $\mathfrak{A}$ has characteristic zero, $\mathfrak{N}$ is the radical of $\mathfrak{A}$ and $\mathfrak{H}$ is the radical of $S(\mathfrak{A})^*$ then it is shown that $S(\mathfrak{N})\subseteq\mathfrak{H}$. From this it is shown that $S(\mathfrak{A})^*$ is semisimple if $\mathfrak{A}$ is semisimple. Jordan modules are defined and the analogue for Jordan algebras of the first Whitehead lemma is proved as a consequence of the result just stated instead of as a lemma used (in the Lie theory) to provide a proof of that result. The paper closes with a proof of the theorem which states that if $\mathfrak{A}$ is a Jordan algebra of characteristic zero and radical $\mathfrak{N}$ then every semisimple subalgebra of $\mathfrak{A}$ can be imbedded in a subalgebra $\mathfrak{S}$ of $\mathfrak{A}$ such that $\mathfrak{A}=\mathfrak{S}+\mathfrak{N}$. *A. A. Albert.*

Schafer, R. D. A theorem on the derivations of Jordan algebras. Proc. Amer. Math. Soc. 2, 290–294 (1951).

Let A be a Jordan algebra over a field F of characteristic 0. It is known [see N. Jacobson, Ann. of Math. (2) **50**, 866–874 (1949); these Rev. **11**, 76] that if A is simple the derivation algebra of A is semisimple or (0), except in the case where A is of dimension 3 over F. It follows at once that the derivation algebra of a semisimple Jordan algebra none of whose simple components is 3-dimensional is semisimple. The author shows here that, conversely, if the derivation algebra of a Jordan algebra A is semisimple or (0) then A is semisimple and has no simple component of dimension 3. The proof utilizes the analogue for Jordan algebras of Wedderburn's principal theorem, which was proved recently by A. J. Penico [see the following review] and reduces the problem to the known associative case by showing first that if the derivation algebra of A is semisimple the radical of A is contained in its (associative) "center" C and that every derivation of C is extensible to a derivation of A.

G. Hochschild (New Haven, Conn.).

Penico, A. J. The Wedderburn principal theorem for Jordan algebras. Trans. Amer. Math. Soc. **70**, 404–420 (1951).

The Wedderburn principal theorem (for algebras of characteristic zero) states that if $\mathfrak{N}$ is the radical of an algebra $\mathfrak{A}$ there exists a subalgebra $\mathfrak{S}$ of $\mathfrak{A}$, necessarily isomorphic to $\mathfrak{A}-\mathfrak{N}$, such that $\mathfrak{A}=\mathfrak{S}+\mathfrak{N}$. The reviewer has proved this result for special Jordan algebras and the result is now proved for arbitrary Jordan algebras. The generalization involves much more than the fact that the exceptional simple Jordan algebra must be handled. In particular, the standard procedure of reducing to the case $\mathfrak{N}^2=0$ is achieved in this case by the derivation of some new results on the ideals of Jordan algebras, in particular, the result that $\mathfrak{A}\mathfrak{N}^2+\mathfrak{N}^2$ is an ideal of $\mathfrak{A}$ properly contained in $\mathfrak{N}$.

A. A. Albert (Chicago, Ill.).

Dubisch, Roy, and Perlis, Sam. On total nilpotent algebras. Amer. J. Math. **73**, 439–452 (1951).

Denote by $\mathfrak{F}^{(n)}$ the total nilpotent algebra of degree n over $\mathfrak{F}$, i.e. any isomorphic copy of the algebra $\mathfrak{F}_n$ of $n\times n$ matrices over $\mathfrak{F}$ with zeros on and above the diagonal. It is shown that these algebras together with their subalgebras constitute the totality of (associative) nilpotent algebras, and that for every nilpotent subalgebra $\mathfrak{N}$ of $\mathfrak{F}_n$ there is an automorphism of $\mathfrak{F}_n/\mathfrak{F}$, carrying $\mathfrak{N}$ into a subalgebra of $\mathfrak{F}^{(n)}$. Stronger results have been previously obtained by G. Köthe [Math. Ann. **103**, 359–363 (1930)] and the reviewer [ibid. **105**, 620–627 (1931)]. The authors then proceed to determine the totality of the automorphisms of $\mathfrak{F}^{(n)}$. An automorphism T of any associative ring $\mathfrak{A}$ is called monic, if $x^T-x\epsilon\mathfrak{A}^{r+1}$ whenever $x\epsilon\mathfrak{A}^r$. It is shown that if $\mathfrak{A}$ is nilpotent, then the totality $\mathfrak{M}$ of such automorphisms forms a normal nilpotent subgroup of the automorphism group $\mathfrak{G}$ of $\mathfrak{A}$. The correspondence $x\to x+bx+xa+bxa$, where $a+b+ba=0$ defines an "inner" automorphism. A monic automorphism N is called "nil", if $u^N=u$ for every u such that either $u\mathfrak{A}=0$ or $\mathfrak{A}u=0$. The totalities $\mathfrak{J}$ and $\mathfrak{N}$ of all inner and all nil automorphisms respectively, are normal subgroups of $\mathfrak{M}$, and for $\mathfrak{A}=\mathfrak{F}^{(n)}$ one has $\mathfrak{M}=\mathfrak{J}\times\mathfrak{N}$. Each diagonal regular matrix d in the algebra $\mathfrak{T}_n$ of all matrices in $\mathfrak{F}_n$ with zeros above the diagonal, determines an automorphism $x\to dxd^{-1}$ in $\mathfrak{T}_n$, which induces a "diagonal" automorphism of $\mathfrak{F}^{(n)}$. These automorphisms form a subgroup $\mathfrak{D}$ of $\mathfrak{G}$, and one has $\mathfrak{G}=\mathfrak{D}\mathfrak{M}=\mathfrak{M}\mathfrak{D}$, $\mathfrak{M}\cap\mathfrak{D}=I$. In the final sections of the paper the authors determine the totality of the ideals in $\mathfrak{F}^{(n)}$, as well as the totality of the characteristic ideals ($\mathfrak{C}$ is a characteristic ideal if $\mathfrak{C}^T=\mathfrak{C}$ for any $T\epsilon\mathfrak{G}$). The study of the latter is reduced in some extent to the case of the so called "elementary" nilpotent algebras. All total nilpotent algebras are e.g. elementary.

J. Levitzki (Jerusalem).

Etherington, I. M. H. Non-commutative train algebras of ranks 2 and 3. Proc. London Math. Soc. (2) **52**, 241–252 (1951).

Train algebras are a class of nonassociative algebras which arose in the author's study of the algebras occurring in genetics. Some of the author's results on commutative train algebras of ranks 2 and 3 [J. London Math. Soc. **15**, 136–149 (1940); these Rev. **2**, 121] are extended to non-commutative train algebras. For train algebras of right and left rank 3, special attention is paid to the "logarithmetic" of the algebra (the arithmetic of the indices of powers of the general element), in which connection see the author's paper on nonassociative arithmetics [Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 442–453 (1949); these Rev. **10**, 677].

R. D. Schafer (Philadelphia, Pa.).

Gotô, Morikuni. On the replicas of nilpotent matrices. Proc. Japan Acad. **23**, no. 5, 39–41 (1947).

The reviewer has proved that every replica of a nilpotent matrix Z with coefficients in a field K of characteristic 0 is of the form tZ, t a scalar [Amer. J. Math. **65**, 521–531 (1943); these Rev. **5**, 171]. This result has been generalized by Tuan, who proved that, if K is of characteristic $p>0$, then every replica Z' of the nilpotent Z may be written in the form of a linear combination of $Z, Z^p, Z^{p^2}, \cdots$ [Bull. Amer. Math. Soc. **51**, 305–312 (1945); these Rev. **7**, 3; a simpler proof has been given by I. S. Cohen, ibid. **52**, 175–177 (1946); these Rev. **7**, 237]. Moreover, Tuan and Cohen proved the result under a weaker assumption than the one that Z' is a replica of Z. If $Z_{r,s}$ represents the effect of the matrix Z on the tensors of type (r, s), Tuan proved that it is sufficient to assume that $Z'=q(Z)$, $Z'_{1,1}=r(Z_{1,1})$ and $Z'_{0,2}=s(Z_{0,2})$, q, r, and s being polynomials with constant term zero, and Cohen that it is sufficient to assume that $Z'=q(Z)$, $Z'_{0,2}=s(Z_{0,2})$ (q and s with constant term 0). In the present paper, the author shows that the following assumptions suffice: if we denote, for any matrix A, by $n(A)$ the set of vectors (of the space on which A operates) which are mapped upon 0 by A, then it is sufficient to assume that $n(Z')\supset n(Z)$, $n(Z'_{0,1})\supset n(Z_{0,1})$, $n(Z'_{0,2})\supset n(Z_{0,2})$.

He uses his result to prove a theorem which implies in particular that a Lie algebra of nilpotent matrices over a field of characteristic 0 is determined by its invariants. This implies a result of Ado [Bull. Soc. Phys.-Math. Kazan (3) **7**, 3–43 (1934)]: Let $\mathfrak{L}$ be a linear Lie algebra over a field of characteristic 0 and $\mathfrak{N}$ an ideal of $\mathfrak{L}$ composed of nilpotent matrices; then $\mathfrak{L}/\mathfrak{N}$ has a faithful linear representation.
C. Chevalley (New York, N. Y.).

Matsushima, Yozô. Note on the replicas of matrices. Proc. Japan Acad. **23**, no. 5, 42–49 (1947).

This paper contains new proofs of the main results in the theory of replicas of a matrix [cf. Chevalley, Amer. J. Math. **65**, 521–531 (1943); these Rev. **5**, 171]. The proofs assume the basic field K to be algebraically closed. It is shown that any matrix A is representable in the form $A_0+\lambda_1A_1+\cdots+\lambda_kA_k$, where A_0 is nilpotent, $\{\lambda_1,\cdots,\lambda_k\}$ is a base of the space spanned over the primitive subfield P of K by the characteristic roots of A, and $A_1,\cdots,A_k$ are semisimple replicas of A whose characteristic roots are in P. The replicas of A are then the matrices $A_0'+\mu_1A_1+\cdots+\mu_kA_k$, where A_0' is any replica of A_0 and $\mu_1,\cdots,\mu_k$ are arbitrary elements of K (this result is new). Considering matrices as endomorphisms of a space M, let N be a subspace of M which is mapped into itself by a matrix A; then it is proved that the replicas of the restriction of A to N are exactly all the restrictions to N of replicas of A. *C. Chevalley.*

Matsushima, Yozô. On the Cartan decomposition of a Lie algebra. Proc. Japan Acad. **23**, no. 5, 50–52 (1947).

The author proves that, if $\mathfrak{L}$ is the Lie algebra of a Lie group G, then any two Cartan subalgebras of $\mathfrak{L}$ may be transformed into each other by an operation of the adjoint group of G. This theorem had been previously established by the reviewer [Amer. J. Math. **63**, 785–793 (1941); these Rev. **4**, 2]; the paper which contained this proof was not available to the author, who refers to it in a note added in proof. *C. Chevalley* (New York, N. Y.).

Kuročkin, V. M. The representation of Lie rings by associative rings. Mat. Sbornik N.S. **28(70)**, 467–472 (1951). (Russian)

It is shown that any Lie ring with operators Ω has a faithful Ω-representation in an associative ring. The case where Ω is a field is the classical result of Birkhoff and Witt. The method used here is similar, but the elements of Ω must be treated with greater care. *I. Kaplansky.*

Theory of Groups

Teissier, Marianne. Sur les équivalences regulières dans les demi-groupes. C. R. Acad. Sci. Paris **232**, 1987–1989 (1951).

Let D be a semigroup, and A a subset of D. Then the minimum congruence E_A for which A is indivisible, i.e., contained in a congruence class, is defined as follows. Let D^* be the smallest semigroup containing D with an identity. Then $a\equiv b\ (E_A)$ if either, (i) neither a nor b belong to D^*AD^*, or (ii) there exists a sequence of complexes $x_1Ay_1,\cdots,x_nAy_n$ $(x_i,y_i\epsilon D^*)$ such that consecutive complexes meet, and $a\epsilon x_1Ay_1$, $b\epsilon x_nAy_n$. The maximum congruence M_A for which A is saturated, i.e., a sum of congruence classes is defined by $a\equiv b\ (M_A)$ if $xay\epsilon A$ if and only if $xby\epsilon A$ $(x,y\epsilon D^*)$. From the above it follows that A is a possible congruence class if and only if when $xAy\cap A\neq 0$, then $xAy\subseteq A$ (i.e., $M_A\supseteq E_A$). The subset A is then a congruence class for all congruences E such that $M_A\supseteq E\supseteq E_A$. Finally, this congruence is unique (i.e., $M_A=E_A$) if the element a in $D'=D/E_A$ corresponding to the class containing A satisfies the condition that if b, c are distinct elements of D'^* and $a\epsilon D'^*bD'^*$, then there exist x, y in D'^* such that $xby=a$ but $xcy\neq a$. *D. Rees.*

Stoll, R. R. Homomorphisms of a semigroup onto a group. Amer. J. Math. **73**, 475–481 (1951).

Let S be a semigroup and let (G,α) denote the pair consisting of a homomorphism α of S onto a group G and G itself. Two such pairs (G,α), (H,β) are termed equivalent if there exists an isomorphism η of G onto H such that $\beta=\alpha\eta$. If η is a homomorphism, we write $(G,\alpha)>(H,\beta)$, and clearly this determines a partial order of the classes of equivalent (G,α)'s. Dubreil [Mém. Acad. Sci. Inst. France (2) **63**, no. 3 (1941); these Rev. **8**, 15] has proved that the classes of equivalent (G,α)'s are in one-to-one correspondence with the normal unitary subsemigroups of S, these being the subsemigroups N satisfying: (a) $ab\epsilon N$ implies $ba\epsilon N$; (b) a, $ab\epsilon N$ implies $b\epsilon N$; (c) $xS\cap N\neq 0$ for all x in S. The author of the present paper shows that this correspondence is an antiisomorphism between the partially ordered set of classes of equivalent (G,α)'s and the partially ordered set of normal unitary subsemigroups of S (ordered by inclusion). He discusses the maximal classes of (G,α)'s or, what is equivalent, the minimal normal unitary subsemigroups, of S. He demonstrates the existence of, and determines, a unique maximal class of (G,α)'s in the following cases: (i) regular sets of partial transformations of a set [Rees, J. London Math. Soc. **22**, 281–284 (1947); these Rev. **9**, 568], (ii) completely simple semigroups without zero [Rees, Proc. Cambridge Philos. Soc. **36**, 387–400 (1940); these Rev. **2**, 127], (iii) semigroups with zeroid elements [Clifford and Miller, Amer. J. Math. **70**, 117–125 (1948); these Rev. **9**, 330]. The related problem where G in (G,α) is taken to be a group with zero is also considered. This corresponds to the consideration of pseudonormal unitary subsemigroups of S, in which condition (c) above does not hold. *D. Rees.*

Hughes, N. J. S. The structure and order of the group of central automorphisms of a finite group. Proc. London Math. Soc. (2) **52**, 377–385 (1951).

Let G be a finite group, C its centre, Q its commutator subgroup, $S_1, S_2, \cdots, S_r$ its Sylow subgroups, and C_i the centre of S_i. The author proves that the group of all central automorphisms of G is isomorphic to the direct product of the groups $\mathfrak{A}_{C_i,R_i}(S_i)$, $i=1,2,\cdots,r$, consisting of those central automorphisms of S_i which leave invariant all elements of the subgroup $R_i=Q\cap S_i$. A similar result holds for the semigroup of central endomorphisms. The structure of the groups $\mathfrak{A}_{C_i,R_i}(S_i)$ is investigated and a formula for their orders is obtained. *D. C. Murdoch* (Vancouver, B. C.).

Azleckiĭ, S. P. On the generation of a finite group by means of a system of Sylow classes. Mat. Sbornik N.S. **28(70)**, 461–466 (1951). (Russian)

Let G be a group of order n, where $\log n=\sum a_i\log p_i$, p_i distinct primes $(i=1,2,\cdots,k,\ k>1)$. By $\langle P_i\rangle$ is meant the collection of Sylow subgroups corresponding to p_i. A system of several $\langle P_i\rangle$ which together generate G is a Sylow system; such a system is minimal if no other system has fewer $\langle P_i\rangle$. The number of $\langle P_i\rangle$ in a minimal system is the

Sylow rank of G. The author considers chiefly the question when a group has just one minimal system. Special groups, solvable groups, and symmetric groups all have this property (*); so does every group with Sylow rank $k-1$. Symmetric groups are generated by their transpositions, and so have Sylow rank 1. The minimal system of a solvable group consists of those $\langle P_i\rangle$ which are not in $G'=[G, G]$. Using Burnside's theorem, it is proved that if $k\leqq 3$, G has property (*) if and only if $G\neq G'$. This is false if $k>3$. The group G is generated by the collection of those $\langle P_i\rangle$ not in G' if the latter is in every maximal subgroup of G. The maximal subgroup M of G is invariant if and only if M contains $k-1$ of the p_i. *J. L. Brenner* (Pullman, Wash.).

Čunihin, S. A. Sylow properties and semi-invariant subgroups. Doklady Akad. Nauk SSSR (N.S.) **77**, 973–975 (1951). (Russian)

Let the terminology and notation be as described earlier [Mat. Sbornik N.S. **25**(67), 321–346 (1949); same Doklady (N.S.) **66**, 165–168 (1949); **69**, 735–737 (1949); **73**, 29–32 (1950); these Rev. **11**, 495; **10**, 678; **11**, 321; **12**, 156]. A Π-permutable subgroup of $\mathfrak{G}$ which commutes with every element in $\mathfrak{G}$ whose order is a power of any prime dividing g/m is called Π-semi-invariant in $\mathfrak{G}$. A finite chain of subgroups from $\mathfrak{G}$ to the identity subgroup $\mathfrak{E}$ such that each term of the chain (after the first) is Π-semi-invariant in the preceding term is called a Π-semi-invariant series of $\mathfrak{G}$. The finite group $\mathfrak{G}$ is of type $\Pi-2$ if and only if $\mathfrak{G}$ possesses a Π-semi-invariant series every factor-structure of which is of type $\Pi-2$. *R. A. Good* (College Park, Md.).

Reed, Irving S. A general isomorphism theorem for factor groups. Math. Mag. **24**, 191–194 (1951).

The author states and proves a generalisation of Zassenhaus' lemma to the case where we are given n pairs of subgroups U_i, X_i of a group, X_i being normal in U_i; and applies it to prove a theorem of Remak [J. Reine Angew. Math. **166**, 65–100 (1931)]. *G. Higman* (Manchester).

Iino, Riichi. On the generalization of nilpotent groups. Mem. Fac. Sci. Eng. Waseda Univ. **14**, 1–2 (1950).

Fedorov, Yu. G. On infinite groups of which all nontrivial subgroups have a finite index. Uspehi Matem. Nauk (N.S.) **6**, no. 1(41), 187–189 (1951). (Russian)

The only group satisfying the conditions in the title of the paper is the infinite cyclic group. As a corollary, every torsion-free group containing at least one cyclic subgroup of finite index is itself cyclic. *R. A. Good.*

Tuan, Hsio-Fu. A theorem about p-groups with abelian subgroups of index p. Acad. Sinica Science Record **3**, 17–23 (1950). (English. Chinese summary)

It is shown that if a group G of prime power order p^n contains a maximal normal Abelian subgroup A with cyclic factor group G/A and if Z is the center of G and K the commutator subgroup of G, then $A/Z\cong K$. In particular, this applies when G has a normal Abelian subgroup of order p^{n-1}. Let the lower central series of G be given by $G=G_1\supset K=G_2\supset\cdots\supset G_c\supset G_{c+1}=I$ and the upper central series by $G=Z_c\supset Z_{c-1}\supset\cdots\supset Z_1=Z\supset Z_0=I$. If the assumption is the same as above, it is shown that $A/(A\cap Z_i)\cong G_{i+1}$ for $i=1, 2, \cdots, c$.

R. Brauer (Ann Arbor, Mich.).

Neumann, Hanna. On an amalgam of abelian groups. J. London Math. Soc. **26**, 228–232 (1951).

An example is given of five Abelian subgroups of a group G which cannot be imbedded into an Abelian group with the same amalgamations as in G. It is shown furthermore that three Abelian subgroups of a group can always be imbedded into an Abelian group with the same amalgamations as before. *R. Baer* (Urbana, Ill.).

Hamstrom, Mary-Elizabeth. Linear independence in Abelian groups. Proc. Amer. Math. Soc. **2**, 487–489 (1951).

For an Abelian group G, and an integer $m=0$ or $m>1$, the rank $r(m, G)$ is defined as the largest number of elements $g_1, \cdots, g_r$ in G such that $\sum c_i g_i=0$ implies all $c_i\equiv 0 \pmod m$. It is noted that the formula $r(G)\geqq r(K)+r(G-K)$ of Alexandroff and Hopf [Topologie, v. 1, Springer, Berlin, 1935] does not hold in general for $m\neq 0$, and a correct proof is given that the opposite weak inequality holds when m is a prime. Theorem 1 evaluates $r(m, G)$, where G is of type $(n_1, \cdots, n_t)$, as the minimum for p^α dividing m of the number of n_j divisible by p^α. Correction: The first sentence on p. 488 should read: "If any $r_i y_i=0$, we are done."

R. C. Lyndon (Princeton, N. J.).

Higman, Donald G. Lattice homomorphisms induced by group homomorphisms. Proc. Amer. Math. Soc. **2**, 467–478 (1951).

Zappa's results [Giorn. Mat. Battaglini (4) **2**(78), 182–192 (1949); these Rev. **11**, 322] on characterizations of group homomorphisms which induce lattice-homomorphisms and groups which admit such homomorphisms, and on decompositions of such groups, are here generalized to infinite groups; it is required that infinite as well as finite meet and join be preserved. For example: Let H be a proper subgroup of a group G, and ω an endomorphism of G such that $S\omega=S\cap H$ for every $S\subset G$; such an ω exists if and only if every element in G has finite order, H is a direct factor of G, and the orders of the elements in H are prime to the orders of the elements in G/H. A group G (and the pair N, G) is said to have the property (Z) if for some normal $N\subset G$, the natural homomorphism ω of G onto G/N induces a proper lattice homomorphism of G onto G/N. Hereafter N denotes a proper normal subgroup of G. Denote by ϕ the set of all primes which are orders of elements in G/N, by K the set of all elements of G whose orders have prime factors exclusively from ϕ, and by N'' the set of all elements of N whose orders have no prime factors from ϕ. Theorem: The pair N, G has property (Z) if and only if the elements of G have finite order, and for each prime p in ϕ, (a) every element in G of order a power of p is in C (the centralizer of N in G), and (b) N_p (set of all elements of N of order a power of p) is contained in the cyclic group generated by any element in G, not in N, having order a power of p. Suppose N is in the center of G, and the existence in N of an element of prime order p implies the existence in G/N of a coset of order p; then N, G has property (Z) if and only if the elements of G have finite order, and for each prime p, every (cyclic) p-group in G contains, or is contained in, N_p. If N, G has property (Z) and G/Z is finite (Z denotes the center of N), then $G=N''\times K$. *P. M. Whitman.*

Zappa, Guido. Sulla condizione perchè un emitropismo inferiore tipico tra due gruppi sia un omotropismo. Giorn. Mat. Battaglini (4) **4**(80), 80–101 (1951).

Terminology: A lattice-homomorphism of the subgroups of a group G onto those of a group G' is called a homotropism;

if permanence of the intersection (union) is omitted from the postulates, then the mapping is called an upper (lower) hemitropism. [This corresponds to the terms join-homomorphism (meet-homomorphism) given by Birkhoff, Lattice Theory, Amer. Math. Soc. Colloquium Publ., v. 25, rev. ed., New York, N. Y., 1948, p. 21; these Rev. 10, 673.] The ordinary homomorphism between two groups is always an upper, but not necessarily a lower hemitropism; a hemitropism induced by an ordinary homomorphism is called typical. If G' is a subgroup of G and any subgroup A of G is mapped onto $A \wedge G'$, then the mapping is always a lower, but not necessarily an upper hemitropism; a lower hemitropism produced in this manner is called typical. In a previous paper [Giorn. Mat. Battaglini (4) 2(78), 182–192 (1949); these Rev. 11, 322] the author has given necessary and sufficient conditions for a typical upper hemitropism between two finite groups to be also a lower hemitropism. In the present paper he solves the corresponding problem: to find necessary and sufficient conditions for a typical lower hemitropism between two finite groups G and $N \subset G$ to be also an upper hemitropism (not necessarily typical) and hence a homotropism. Let g and n be the orders of G and N; let m be the product of those prime factors of g which are coprime to n; l the product of those which are coprime to g/n; h the product of those which divide both n and g/n so that $g=mhl$, $n=kl$, where $h=\prod p_i^{\alpha_i}$, $k=\prod p_i^{\beta_i}$, $0<\beta_i<\alpha_i$. Then the typical lower hemitropism between G and N is also an upper hemitropism if and only if (i) N is normal in G; (ii) $G=MHL$, $N=KL$, with M, H, K, L of order m, h, k, l; (iii) M and L are characteristic in G; (iv) H is cyclic or the direct product of a generalised quaternion group with a cyclic group of odd order; (v) K is cyclic and, in case H is not cyclic, the order of K is twice an odd number; (vi) if S_i is a Sylow subgroup of H, then every subgroup of L which is permutable with all elements of $S_i \wedge N$ is permutable with all elements of S_i. The main tool of the proofs is the distributive law $(A \vee B) \wedge N=(A \wedge N) \vee (B \wedge N)$ which holds for subgroups A, B, N of G, if and only if the typical lower hemitropism between G and N is also an upper hemitropism. There are some connections between the author's investigations and those of S. Čunihin on Π-properties of finite groups [see e.g. Mat. Sbornik N.S. 25(67), 321–346 (1949); these Rev. 11, 495].
K. A. Hirsch (London).

Dinkines, Flora. Semi-automorphisms of symmetric and alternating groups. Proc. Amer. Math. Soc. 2, 478–486 (1951).

Un semi-automorphisme d'un groupe G est une application biunivoque $a \rightarrow a'$ de G sur lui-même laissant invariant l'élément neutre e de G et telle que $(aba)'=a'b'a'$ indentiquement. L'auteur dit qu'un groupe de permutations d'un ensemble E est un groupe symétrique (resp. alterné) s'il est formé de toutes les permutations (resp. de toutes les permutations paires) lorsque E est fini, et de toutes les permutations déplaçant au plus un sous-ensemble de E de puissance majorée par un cardinal infini (resp. des permutations paires ne déplaçant qu'un nombre fini d'éléments) si E est infini. Elle démontre que tout semi-automorphisme d'un groupe symétrique ou alterné est un automorphisme ou un anti-automorphisme (isomorphisme sur le groupe opposé). La méthode consiste à montrer que toute permutation d'un de ces groupes est de la forme aba, où a et b sont d'ordre 2 ou 3, puis à prouver qu'il existe un automorphisme ou anti-automorphisme φ tel que $\varphi(x')=x$ lorsque x est d'ordre 2 ou 3.
J. Dieudonné (Nancy).

Nagao, Hirosi. On the theory of representation of finite groups. Osaka Math. J. 3, 11–20 (1951).

New proofs are given for the orthogonality relations for the characters of a group of finite order, for the theorems concerning the Kronecker products of two representations, and for the reciprocity theorems for induced representations [cf. the reviewer and Nesbitt, University of Toronto Studies, Mathematical Series, no. 4, 1937; Ann. of Math. (2) 42, 556–590 (1941); these Rev. 2, 309; T. Nakayama, ibid. 39, 361–369 (1938)]. The results apply as well to the ordinary as to the modular theory and they can also be used for the study of the representations of the group by collineations as given by K. Asano and K. Shoda [Compositio Math. 2, 230–240 (1935)] and K. Asano, M. Osima, and M. Takahasi [Proc. Phys.-Math. Soc. Japan (3) 19, 199–209 (1937)].
R. Brauer (Ann Arbor, Mich.).

Hewitt, Edwin, and Zuckerman, H. S. A group-theoretic method in approximation theory. Ann. of Math. (2) 52, 557–567 (1950).

The main result of this paper is Theorem 1: Let S be any group, and let Σ be any group of characters of S (i.e. homomorphisms of S into the multiplicative group T of complex numbers of absolute value 1). Then, if χ is any character of Σ, if $\sigma_1, \sigma_2, \cdots, \sigma_m$ are arbitrary elements of Σ, and if ϵ is any positive real number, there exists an element $s \epsilon S$ such that

$$|\chi(\sigma_j)-\sigma_j(s)|<\epsilon \quad (j=1, 2, \cdots, m).$$

The proof makes use of Zorn's lemma, of the Weierstrass-Stone approximation theorem, and, very essentially of a theorem of T. Tannaka [Tôhoku Math. J. 45, 1–12 (1938)] and M. Krein [C. R. (Doklady) Acad. Sci. URSS (N.S.) 30, 9–12 (1941); these Rev. 2, 316] which asserts the following. Let $\mathfrak{T}$ be the set of all finite linear combinations (with complex coefficients) of the (irreducible continuous) representation coefficients of a topological group G. If H is an algebra-homomorphism of $\mathfrak{T}$ into the complex number system, such that $H(f\bar{f})\geqq 0$ for any $f \epsilon \mathfrak{T}$, then H is continuous in the uniform topology for $\mathfrak{T}$ and can therefore be extended continuously over the uniform closure of $\mathfrak{T}$ in the algebra of all continuous functions on G. Theorem 2 is a consequence of Theorem 1: Every character χ of a locally compact Abelian group G is the pointwise limit of continuous characters, in the sense that, for every $\epsilon>0$ and every finite subset $g_1, \cdots, g_m$ of elements of G, there is a continuous character τ of G such that

$$|\chi(g_j)-\tau(g_j)|<\epsilon \quad (j=1, 2, \cdots, m).$$

These theorems are first applied to derive some classical approximation theorems, such as Kronecker's (Theorems 3–5). Here, and often in the sequel, a Hamel basis construction is also used. Next, it is proved (Theorem 6) that a real linear topological space X, when considered as an Abelian group (with respect to vector-addition), has only the continuous characters $\varphi(x)=e^{2\pi i f(x)}$, where $f(x)$ is a (uniquely determined) continuous linear functional on X. From this result and from Theorem 1 one gets Theorem 7: Let $y_1, y_2, \cdots, y_m$ be rationally independent continuous linear functionals on the real linear topological space X, and let $\alpha_1, \alpha_2, \cdots, \alpha_m$ be arbitrary real numbers. Then, for any $\epsilon>0$, it is possible to find an $x \epsilon X$ such that

$$|\alpha_j-y_j(x)|<\epsilon \ (\mathrm{mod}\ 1) \quad (j=1, 2, \cdots, m).$$

In the particular case $X=C(0, 1)$, this theorem may be formulated as follows (Theorem 11): Let $g_1, g_2, \cdots, g_m$ be rationally independent, real-valued functions of bounded

variation of the interval [0, 1]. If $\alpha_1, \alpha_2, \cdots, \alpha_m$ are any real numbers and $\epsilon>0$, then there exists a continuous real-valued function $f(x)$ on [0, 1] such that

$$\left|\alpha_j-\int_0^1 f(x)dg_j(x)\right|<\epsilon \pmod 1 \qquad (j=1, 2, \cdots, m).$$

As further applications, some theorems are derived on approximation properties of p-adic numbers. The various examples, classical and new, illustrate the range of the present method. Some possible ways of generalization are pointed out. *B. Sz.-Nagy* (Szeged).

Goodstein, R. L. An introduction to the theory of continuous groups. Math. Gaz. 35, 91–96 (1951).

Expository paper.

Matsushima, Yozô. On the discrete subgroups and homogeneous spaces of nilpotent Lie groups. Nagoya Math. J. 2, 95–110 (1951).

This paper begins with a theorem on the situation of a discrete subgroup D of a connected simply connected solvable Lie group G, which is then used as one of the tools for studying nilpotent Lie groups and their homogeneous spaces. Some of the results have been obtained by Mal'cev [Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 9–32 (1949); these Rev. 10, 507]. A homogeneous space M of a connected nilpotent Lie group is homeomorphic to the product of a compact space C and a Euclidean space. A certain closed subgroup of G is transitive on C. If M is a compact homogeneous space of a connected nilpotent Lie group then $M=G/D$ where D is discrete and where G is a connected simply connected nilpotent rational Lie group. A Lie group is called rational if the structure constants are rational in some coordinate system. Two compact homogeneous spaces of nilpotent groups are homeomorphic if they have isomorphic fundamental groups. The last part of the paper is concerned with homology properties. The rational cohomology groups of dimension one and two of a compact homogeneous space of a connected, simply connected nilpotent Lie group G are the same as the cohomology groups of the Lie algebra of G. *D. Montgomery.*

Kuranishi, Masatake. On everywhere dense imbedding of free groups in Lie groups. Nagoya Math. J. 2, 63–71 (1951).

A locally compact group G is said to be approximated by discrete groups if there is a sequence D_n of discrete subgroups such that every open set in G contains a point in almost all D_n. If a Lie group G can be approximated by discrete groups, then G is nilpotent and rational. There are, however, rational nilpotent groups which can not be approximated by discrete subgroups. If G is a connected Lie group a necessary and sufficient condition for G to be approximated by discrete groups is that G contains a discrete group D satisfying the following condition: There is a neighborhood U of e which can be identified with a neighborhood of the corresponding Lie algebra L and $U\cap D$ contains a basis of L. There are interconnections between this paper and the paper reviewed above and connections with the Mal'cev paper cited there. The author next considers Lie algebras which are generated by two elements. He shows that a semisimple Lie algebra is also so generated. This is used to prove that connected semisimple Lie groups, and also certain other groups, contain a dense subgroup isomorphic to a free group generated by two elements. *D. Montgomery* (Princeton, N. J.).

Chevalley, Claude. Sur le groupe exceptionnel (E_6). C. R. Acad. Sci. Paris 232, 1991–1993 (1951).

E. Cartan has exhibited a representation of the exceptional Lie group (E_6) as the group of linear transformations which keep a certain cubic form F in 27 variables invariant. In this paper, the author gives a new interpretation to F and generalizes Cartan's result to Lie algebras of type (E_6) over any algebraically closed field K of characteristic 0. The new definition of F is as follows: Let V be a 6-dimensional linear space over K, and let E_p denote the subspace of homogeneous elements of degree p of the exterior algebra E of V. Define $W=V\times V\times E_4$, and for $w=(x, y, a)\epsilon W$, define $F(W)=6xy\vee a+a\vee a\vee a$, where "$\vee$" denotes the regressive intersection product in E. The author proves that the algebraic Lie group G consisting of all the linear transformations of W which keep F invariant is of type (E_6). The cone $\mathfrak{F}$ with equation $F=0$ as well as the variety $\mathfrak{S}$ of multiple points of $\mathfrak{F}$ are invariant under G. The variety $\mathfrak{S}$ is the cone defined by the equations $xa=0$, $ya=0$, $2xy+a\vee a=0$. G is transitive on the hypersurface with equation $F=c\neq0$, as well as on the nonzero points of $\mathfrak{S}$. *G. D. Mostow.*

Chevalley, Claude. Sur une variété algébrique liée à l'étude du groupe (E_6). C. R. Acad. Sci. Paris 232, 2168–2170 (1951).

The author continues the analysis begun in the paper reviewed above [see that review for notation]. The author denotes $V^*\times V^*\times E^*$ (V^* and E^* being the duals of V and E) by W^* and defines W^* as dual to W by means of the bilinear form $\langle w, w^*\rangle=\langle x, x^*\rangle-\langle y, y^*\rangle+\langle a, a^*\rangle$, where

$$w=(x, y, a)\epsilon W, \quad w^*=(x^*, y^*, a^*)\epsilon W^*.$$

$F^*(W^*)=6x^*y^*\vee a^*+a^*\vee a^*\vee a^*$ is then invariant under $^ts^{-1}$ for $s\epsilon G$. Let J be any isomorphism of the algebra E onto E^*, and J_1 denote the isomorphism $(x, y, a)\rightarrow(Jx, Jy, Ja)$ of W onto W^*. If $Je=e^*$, where e, e^* are the distinguished elements of E_6 and E_6^* that were selected in the definition of the regressive intersection product "$\vee$", then J_1 transforms F into F^*. The mapping $s\rightarrow J_1^{-1}{}^ts^{-1}J_1$ is an outer automorphism of G and is the only one, up to inner automorphisms. The author then proceeds to deduce a biunique birational correspondence between a certain family of 4-dimensional linear subspaces on an 8-dimensional projective quadric and a 10-dimensional projective variety in a 15-dimensional projective space. All the varieties in question arise from cones in $\mathfrak{S}$. This correspondence is related to two different representations of a subgroup of G of type (D_5). Erratum: p. 2169, line 15: Read "16" for "26". *G. D. Mostow* (Syracuse, N. Y.).

***Leray, Jean. Sur l'homologie des groupes de Lie, des espaces homogènes et des espaces fibrés principaux.** Colloque de topologie (espaces fibrés), Bruxelles, 1950, pp. 101–115. Georges Thone, Liège; Masson et Cie., Paris, 1951. 175 Belgian francs; 1225 French francs.

The paper deals with the homology properties, with real coefficients, of Lie groups, homogeneous spaces and, more generally, principal fiber bundles. Let X be a principal fiber bundle with the fiber F, G a maximal connected subgroup of F, T a maximal toroid of G, and N the normalisator of T in G. If L_T is the tangent vector space to T at the unit element, the author considers the algebra $\mathcal{P}_T$ of polynomials whose arguments are the coordinates of a point of L_T. The quotient group $\Phi=N/T$ is a finite group, and acts in $\mathcal{P}_T$. The main results of the author are stated in three theorems. The first theorem deals with the subalgebra $\mathcal{P}_G$ of $\mathcal{P}_T$ formed

by all the polynomials of $\mathcal{P}_T$ invariant under Φ. It contains a vector subspace $\mathfrak{Q}_G$ of $\mathcal{P}_G$ of dimension l equal to the rank of F, whose elements generate the elements of positive degree of $\mathcal{P}_G$. Moreover, if $\mathcal{R}_G$ is the minimal ideal of $\mathcal{P}_T$ containing $\mathfrak{Q}_G$, the cohomology algebra $\mathcal{H}_{G/T}$ of the homogeneous space G/T is isomorphic to $\mathcal{P}_T/\mathcal{R}_G$. There is furthermore a canonical isomorphism of $\mathfrak{Q}_G$ onto a vector subspace $\mathcal{V}_G$ of $\mathcal{H}_G$, lowering the degree of each element by 1, such that the exterior algebra of $\mathcal{V}_G$ is isomorphic to the cohomology algebra of G.

The second theorem studies relations between the homology properties of X/U and X/F, where U is a subgroup of F having the same rank. A typical consequence is a conjecture of Hirsch which states that if G is a group manifold with the Poincaré polynomial

$$G_t=(1+t^{2m_1-1})\cdots(1+t^{2m_l-1}),$$

and U a connected subgroup of G having the same rank and with the Poincaré polynomial

$$U_t=(1+t^{2n_1-1})\cdots(1+t^{2n_l-1}),$$

then the Poincaré polynomial of the homogeneous space G/U is

$$(G/U)_t=\frac{(1-t^{2m_1})\cdots(1-t^{2m_l})}{(1-t^{2n_1})\cdots(1-t^{2n_l})}.$$

The third theorem studies the spectral algebra of the projection $G/S\to G/T$, where S is a toroidal subgroup of T, and gives an interpretation of the second group of the sequence.

The results of the paper have contact with recent works of H. Cartan, Chevalley, Koszul, and Weil. The author, however, makes use of his spectral cohomology theory of a mapping.

S. Chern (Chicago, Ill.).

Calabi, Lorenzo. Le estensioni centrali di gruppi. Boll. Un. Mat. Ital. (3) **5**, 264–266 (1950).

This is an extension of earlier work [Calabi and Ehresmann, C. R. Acad. Sci. Paris **228**, 1551–1553 (1949); Calabi, ibid. **229**, 413–415 (1949); these Rev. **11**, 9, 158]. An extension $E(B, F)$ is central if the automorphisms induced in F are all inner. A characterization of central extensions is given in Theorem 1; a corollary says that the group of central extensions of F by B is isomorphic with the second cohomology group $H^2(B, C)$, where C is the center of F, and B operates trivially on C; this is related to and completes results of Eilenberg and MacLane [Ann. of Math. (2) **48**, 326–341 (1947); these Rev. **9**, 7]. The group of extensions entering here has been defined in a special case by A. S. Shapiro [ibid. (2) **50**, 581–586 (1949); these Rev. **11**, 157]. If B and F are topological, one considers the subgroup $\mathcal{K}^2(B, C)$ of $H^2(B, C)$ determined by cocycles which are continuous in the neighborhood of (e, e). The group of topological extensions is isomorphic with $\mathcal{K}^2(B, C)$ if B is connected and F compact or semi-simple Lie, or if B is connected and locally simply connected and F discrete (Theorem 2). The same isomorphism holds (Theorem 3) if B is connected and has a covering $\tilde{B}$, such that all extensions $E(\tilde{B}, F)$ are trivial. The formula of Shapiro describing the group of extensions in terms of the kernel of the map of $\tilde{B}$ onto B and its homomorphisms into C, is still valid in this case (Theorem 3′).

H. Samelson (Ann Arbor, Mich.).

Calabi, Lorenzo. Su alcuni rapporti tra la teoria delle estensioni ed il gruppo degli automorfismi del gruppo esteso. Boll. Un. Mat. Ital. (3) **5**, 286–289 (1950).

[Cf. the preceding review.] Let $A(F)$ and $I(F)$ denote the groups of outer resp. inner automorphisms of F. An extension $E(B, F)$ determines an antihomomorphism of E into $A(F)$; the image X is an extension of $I(F)$ by a certain group B^*. There is an induced antihomomorphism $\dot{\chi}$ of B into B^*. Theorem I states the equivalence of two conditions: (1) There exists an extension $E(A(F)/I(F), F)$ whose $\dot{\chi}$ is the symmetry $(x\to x^{-1})$; (2) there exist extensions $E(B, F)$ with arbitrarily prescribed B and $\dot{\chi}$. An extension is pre-inessential if the induced extension of F/C by B is inessential. Theorem 2 and corollaries give criteria for pre-inessentiality, e.g.: $E(B, F)$ is pre-inessential if $\dot{\chi}$ can be lifted to an antihomomorphism of B into $A(F)$. It follows that $E(B, F)$ is pre-inessential if the corresponding $X(B^*, I(F))$ is inessential; this is certainly so, if $A(F)$ is an inessential extension of $I(F)$. The group of pre-inessential extensions $E(B, F)$ with given $\dot{\chi}$ is isomorphic with $H^2(B, C, \chi)$ (lemma 2). There are more results of the same type, also for topological groups.

H. Samelson (Ann Arbor, Mich.).

Gottschalk, W. H., and Hedlund, G. A. Asymptotic relations in topological groups. Duke Math. J. **18**, 481–485 (1951).

The authors prove the following theorem: Let G be a locally compact Abelian group, let ν be a Haar measure in G, let E be a totally bounded measurable subset of G such that some translate of the interior of E contains 0 and generates G and let $x\epsilon G$; then $\nu((nE)\cap(nE+x))\sim\nu nE\sim\nu(n+1)E$. Here nE denotes the set of all sums of n elements in E and $f_n\sim g_n$ means $\lim_{n\to\infty} f_n/g_n=1$. This is a generalization of a theorem of Y. Kawada, who proved a similar statement under the further assumptions that G is connected and E is open [Proc. Imp. Acad. Tokyo **19**, 264–266 (1943); these Rev. **7**, 240]. Since G is generated by some compact neighborhood of 0, the proof of the theorem is easily reduced to the case where G is a closed subgroup of a finite-dimensional vector group V over the reals. Let A be a compact convex subset of V whose interior is not empty and let x be any element in V, then it is proved, using Kawada's result, that, if we normalize Haar measures of V and G suitably, the measures of the sets nA and $nA\cap(nA+x)$ in V and the measures of the sets $nA\cap G$ and $nA\cap(nA+x)\cap G$ in G are all equivalent, in the above sense. The theorem follows from this lemma by comparing the measures of nE and $nE\cap(nE+x)$ with those of nE' and of $nE'\cap(nE'+x)$, where E' is the convex closure of E in V.

K. Iwasawa.

Krishnan, Viakalathur S. L'équivalence de quelques représentations d'une structure abstraite. C. R. Acad. Sci. Paris **232**, 918–920 (1951).

(1) Es gibt genau dann ein direktes Produkt pseudometrischer Räume über einem vollständigen regulären Raum R, wenn die Topologie von R Produkt von Pseudometriken ist. (2) Es gibt genau dann ein direktes Produkt geordneter Gruppen über eine Verbandsgruppe G, wenn die Halbordnung von G Konjunktion von Ordnungen ist. Verf. skizziert (ohne Beweis), wie beide Sätze (1), (2) aus einem Satz über abstrakte Strukturen abgeleitet werden können.

P. Lorenzen (Bonn).

NUMBER THEORY

Watson, G. L. Pandiagonal and symmetrical magic squares. Math. Gaz. 35, 108–109 (1951).

Gloden, A. Sur quelques congruences d'ordre supérieur. Bull. Soc. Roy. Sci. Liège 19, 429–436 (1950).

The congruence $x^4+1\equiv 0$ is discussed modulo p, p^2, and p^3. It is known to be solvable only for $p=8k+1$. Tables of minimal solutions are given for all $p<1000$ in the mod p^2 case, and for $p<200$ in the mod p^3 case. *I. Niven.*

Moppert, Karl-Felix. Über eine diophantische Identität. Comment. Math. Helv. 25, 71–74 (1951).

The author discusses identities (1) $P_1^{k_1}+P_2^{k_2}+P_3^{k_3}\equiv 0$, where k_1, k_2, k_3 are integers greater than 1, and P_1, P_2, P_3 are relatively prime nonconstant polynomials over the field of complex numbers. He obtains as a necessary condition for the existence of (1) that (2) $k_1^{-1}+k_2^{-1}+k_3^{-1}>1$; the set of solutions of (2) is: $(2, 2, k)$, $k>1$; $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$. For each triple (k_1, k_2, k_3) in the above set he shows that the Schwarz triangle function mapping the circular arc triangle with angles π/k_1, π/k_2, π/k_3 onto the upper half plane—which is a rational function—can be set equal to $P_1^{k_1}/P_3^{k_3}$ and (1) is satisfied. The identity obtained from $(2, 3, 3)$ reads:

$$p(px^4-64qxy^3)^3+q(8px^3y+512qy^4)^3 =(p^2x^6+160pqx^3y^3-512q^2y^6)^2,$$

where the letters appearing are arbitrary complex numbers. By specialization to integers one obtains infinitely many solutions of the diophantine equation $pX^3+qY^3=Z^2$. The remaining identities are treated similarly. *J. Lehner.*

Erdös, P. On a Diophantine equation. J. London Math. Soc. 26, 176–178 (1951).

The equation $\binom{n}{k}=x^j$ with the natural restrictions $n\geqq 2k$, $j>1$, $k>1$ is proved impossible in integers for all $k>3$. The author [same J. 14, 245–249 (1939); these Rev. 1, 39] established the impossibility in case $j=3$, and Obláth [ibid. 23, 252–253 (1948); these Rev. 10, 353] extended this to $j=4$ and $j=5$. It can be readily proved that there are infinitely many solutions in case $j=k=2$, and Erdös recalls that there is only one solution ($n=50$, $x=140$) in case $k=3$, $j=2$, although the reference is forgotten. Thus it appears that the cases $k=2$ and $k=3$ with $j\geqq 6$ remain open. *I. Niven* (Eugene, Ore.).

Venkataraman, C. S. An order result relating to the number of divisors of n in an arithmetic progression. Math. Student 18 (1950), 19–21 (1951).

Let $d(a, b|n)$ denote the number of divisors of n of the form $ar+b$ ($r=0, 1, 2, \cdots$). Then the author proves that there exists a constant $\alpha=\alpha(a, b)$ such that as $x\rightarrow\infty$

$$a\sum_{n\leq x} d(a, b|n)=x\log x+\alpha x+O(x^{\frac{1}{2}}).$$

The proof follows along the lines of Dirichlet's argument in the well-known case of $a=b=1$. It may be worth noting that $\alpha(a, b)=\log a-\psi(b/a)-\psi(2)$ where $\psi(x)$ is the logarithmic derivative of $\Gamma(x)$. *D. H. Lehmer* (Berkeley, Calif.).

Porcelli, Pasquale, and Pall, Gordon. A property of Farey sequences, with applications to qth power residues. Canadian J. Math. 3, 52–53 (1951).

Let n be a positive integer and let t_i/u_i be the ith term of a Farey series. Let $S_n(t_i, u_i)$ be the (possibly vacuous) sequence of integers of the form $(nt_i+a)/u_i$ produced when a ranges over the integers in the interval

$$\frac{-n}{u_{i-1}+u_i}<a\leqq\frac{n}{u_i+u_{i+1}}.$$

Then the authors prove that the sequence of sequences $S_n(t_0, u_0)$, $S_n(t_1, u_1)$, $\cdots$, consists of the sequence $1, 2, \cdots, n$ in this order. This theorem is applied to the problem of solving the binomial congruence $x^q\equiv D \pmod p$. The following result is obtained: Let p be an odd prime, n and q be positive integers, $p>2h-2$, and D denote a qth power residue modulo p. Then one of the numbers Du^q ($u=1, 2, \cdots, h-1$) is congruent modulo p to at least one of the numbers $(\pm t)^q$, $t=1, 2, \cdots, [p/h]$. *D. H. Lehmer.*

Moser, Leo. A theorem on quadratic residues. Proc. Amer. Math. Soc. 2, 503–504 (1951).

Dirichlet's class-number formula implies that, if p is a prime of the form $4n+3$, there are more quadratic residues of p than nonresidues between 0 and $\frac{1}{2}p$. No proof of this theorem has yet been given which does not employ infinite series. The author simplifies the standard proof by using the Fourier series $\sum_{m=1}^{\infty}\sin(2m-1)\theta/(2m-1)$ instead of the customary $\sum_{m=1}^{\infty}\sin m\theta/m$. [To the author's list of related proofs should be added the following: Chowla, Proc. Nat. Inst. Sci. India 13, 231–232 (1947); these Rev. 9, 272.] *A. L. Whiteman* (Los Angeles, Calif.).

Ankeny, N. C., and Rogers, C. A. A conjecture of Chowla. Ann. of Math. (2) 53, 541–550 (1951).

Demonstration of the following theorem: If for all but a finite number of primes p, an integer a is a residue of an nth power (mod p), then either a is the nth power of some integer (Chowla conjectured that such is always the case) or $n\equiv 0 \pmod 8$ and $a=2^{n/2}a'$, where a' is an nth power. This demonstration is based, in fact, on Bauer's theorem that the sets of prime ideals of a field, which are completely decomposed in its two normal extensions, cannot differ only by finite sets, when these extensions are different. This theorem is applied, in taking as basis field the rational field, to the following couples of its extensions: (1) the field K_{2^r} of 2^r-th roots of unity and $K_{2^r}[(2c^2)^{\frac{1}{2}}]$; (2) K_{2^r} and $K_{2^r}(a^{\frac{1}{2}})$, where a has, as contribution of some odd prime p, an odd power of this prime; (3) the field K_q of qth roots of unity, where q is a power of any odd prime, and $K_q(a^{1/q})$, where a is not a qth power. In using Kronecker's density theorem in place of Bauer's theorem, the same result is stated under weaker conditions (that a is a residue of an nth power (mod p) for a set of p of density 1). *M. Krasner* (Paris).

Rosser, J. Barkley. Real roots of real Dirichlet L-series. J. Research Nat. Bur. Standards 45, 505–514 (1950).

It is surprisingly hard to prove for an individual discriminant d that $\sum_{n=1}^{\infty}(d/n)n^{-s}>0$ for $s>0$. The author has previously proved this for all $|d|\leqq 227$ except $d=-163$ [Bull. Amer. Math. Soc. 55, 906–913 (1949); these Rev. 11, 332]. In this paper the result is also proved for $d=-163$ by first showing numerically that the series is positive for $s=\frac{1}{2}$ and then proving that the series can have at most one zero in $\frac{1}{2}\leqq s<1$. The author also discusses a different method due to Chowla and Selberg [Proc. Nat. Acad. Sci. U. S. A. 35, 371–374 (1949); these Rev. 11, 84]. *H. Heilbronn.*

Karamata, J., and Tomić, M. On an inequality of Kuzmin-Landau concerning trigonometric sums and its application to the Gauss sum. Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka **198**, 163–174 (1950). (Serbo-Croatian)

See Acad. Serbe Sci. Publ. Inst. Math. **3**, 207–218 (1950); these Rev. **12**, 482.

Lomadze, G. A. On the representation of numbers by sums of an odd number of squares. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **17**, 281–314 (1949). (Georgian. Russian summary)

In an earlier paper [same Trudy **16**, 231–275 (1948)] the author obtained explicit formulae for the number $r_s(n)$ of representations of n as the sum of s squares, where s is odd and $9 \leqq s \leqq 23$. These formulae involve infinite products of the type

$$\prod_{p \nmid n} \left\{ 1 - \left(\frac{\pm n}{p} \right) p^{-\sigma} \right\}^{-1}, \qquad \sigma = \frac{s-1}{2},$$

where $\left(\frac{a}{b}\right)$ denotes Jacobi's symbol. The object of the present paper is to replace these products by finite expressions. If $n = l^2 t$, where t is squarefree, it is easily seen that the problem reduces to that of expressing the series

$$L(\sigma, \pm t) = \sum_{u=1}^{\infty} \left(\frac{\pm t}{u} \right) u^{-\sigma}$$

in finite form. The method, though elementary, involves a considerable amount of computation, and since the full statement of results proved in the paper takes up some 13 pages it is impossible to quote them here in detail. As an example we may, however, mention the identity

$$r_{13}(n) = \frac{3932160}{691} \pi^{-6} n^{11/2} \chi_2(n) T_6(n) L(6, t) + \frac{6968}{691} \sum (x_1^4 - 3x_1^2 x_2^2);$$

here the summation on the right ranges over sets of integers $x_1, \cdots, x_{13}$ satisfying $x_1^2 + \cdots + x_{13}^2 = n$; $\chi_2(n)$ depends on the residues (mod 8) of γ and m, where $n = 2^\gamma m$, and m is odd; $T_6(n)$ is a rather complicated finite product, and $L(6, t)$ can be expressed, in each of the cases that may arise, as a finite sum. For instance, when $t > 1$, $t \equiv 1 \pmod 4$, we have

$$L(6, t) = -2\pi^6 t^{-\frac{1}{2}} \sum_{0 < h \leqq \frac{1}{2} t} \left(\frac{h}{t} \right) \left\{ \frac{h^2}{2^4 \cdot 3 \cdot t^3} - \frac{h^4}{2^3 \cdot 3 \cdot t^4} + \frac{h^5}{3 \cdot 5 \cdot t^6} \right\}.$$

L. Mirsky (Bristol).

Halberstam, H. Representation of integers as sums of a square of a prime, a cube of a prime, and a cube. Proc. London Math. Soc. (2) **52**, 455–466 (1951).

Davenport and Heilbronn [Proc. London Math. Soc. (2) **43**, 73–104 (1937)] proved that almost all positive integers are the sums of one square and two cubes (of positive integers). It is proved in this paper that almost all positive integers n are representable in the form $n = p_1^2 + p_2^3 + x^3$, where p_1 and p_2 are primes, and x is a positive integer. The proof involves the Hardy-Littlewood technique of Farey dissection. The treatment of the singular series is based upon the method of Davenport and Heilbronn as modified by Roth [J. London Math. Soc. **24**, 4–13 (1949); these Rev. **10**, 431]. *A. L. Whiteman* (Los Angeles, Calif.).

Kubilyus, I. The decomposition of prime numbers into two squares. Doklady Akad. Nauk SSSR (N.S.) **77**, 791–794 (1951). (Russian)

The author's main result is that the number of solutions of $p = a^2 + b^2$, where p is a prime and a, b integers, and where $p \leqq x$, $0 < a \leqq x^{7/16+\epsilon}$, tends to infinity with x. More generally, the number of prime ideals $\tilde{\omega}$ in an imaginary quadratic field K satisfying $N\tilde{\omega} < x$ and $0 < \arg \tilde{\omega} < x^{-1/16+\epsilon}$ tends to infinity with x. This is an extension of Hecke's famous result about the equal distribution of the arguments of prime ideals [Math. Z. **1**, 357–376 (1918); **6**, 11–51 (1920)]. Hecke's zeta-function for the field K is defined by

$$Z(s, m) = g^{-1} \sum_{\alpha} \exp(img \arg \alpha)(N\alpha)^{-s},$$

summed over all integers $\alpha \neq 0$ of K, where m is any rational integer and g denotes the number of units in K. The author states that by using Vinogradov's estimates for Weyl sums, one can prove, following Tchudakoff [Mat. Sbornik N.S. **19**(61), 47–56 (1946); these Rev. **8**, 197], that $Z(s, m) \neq 0$ for

$$|m| \leqq M, \quad 1 - c(\log M)^{-\theta} \leqq \sigma \leqq 1, \quad |t| \leqq (\log M)^2,$$

where $M \geqq 2$. Here c is a positive constant depending only on K, and θ is an absolute constant with $0 < \theta < 1$. He then proves two theorems. Theorem 1: The number of values of m with $|m| \leqq M$ (where $M \geqq 2$) for which $Z(s, m)$ has at least one zero in the rectangle $\omega \leqq \sigma \leqq 1$, $|t| \leqq (\log M)^2$ is $O(M^{(16+\epsilon)(1-\omega)})$. The proof, which is given only in outline, is based on methods of Linnik [Mat. Sbornik N.S. **15**(57), 3–12 (1944); these Rev. **6**, 260]. The constants implied by O depend on K and ϵ. Theorem 2: Suppose that $0 \leqq \varphi_2 - \varphi_1 \leqq 2\pi$. Then

$$\sum_{\varphi_1 < \arg \tilde{\omega} \leqq \varphi_2} \log(N\tilde{\omega}) \exp(-x^{-1} N\tilde{\omega}) = \frac{gx}{2\pi}(\varphi_2 - \varphi_1)(1 + o(1)) + O(x^{15/16+\epsilon}).$$

This gives the main result. *H. Davenport* (London).

Linnik, Yu. V. Some conditional theorems concerning binary problems with prime numbers. Doklady Akad. Nauk SSSR (N.S.) **77**, 15–18 (1951). (Russian) (s)

Assuming the Riemann hypothesis for $\zeta(\theta)$ the author shows that for every large integer N and $\epsilon > 0$ primes p and p' can be found such that $|N - p - p'| < (\log N)^{3+\epsilon}$. The proof is based on the formula

$$\int_0^{(\log N)^{-3-\epsilon}} \left(\sum_p \log p \, e^{-2\pi i p\alpha - p/N} \right)^2 e^{2\pi i N\alpha} d\alpha = \tfrac{1}{2} e^{-1} N + O(N(\log N)^{-\epsilon}).$$

The reviewer is unable to follow the details of the rather sketchy proof. *H. Heilbronn* (Bristol).

Chevallier, Jean-Maurice. Méthodes, problèmes et résultats nouveaux concernant la répartition des nombres premiers. Rev. Gén. Sci. Pures Appl. N.S. **57**, 102–106 (1950).

The author puts the methods of the sieve of Eratosthenes into the form of a system of difference equations, after having transformed the multiplicative problem into an additive one by using the logarithms of the prime numbers and the integers. (He discusses also generalized prime numbers and generalized integers.) The solution of this system of difference equations could be obtained step by step as in the Eratosthenes method itself. For a general solution, however, the author submits only an approximation, based

moreover on a hypothesis. After that also some probabilistic arguments are used. The whole presentation is not precise enough to permit a final evaluation of the methods set forth in this paper. *H. Rademacher* (Philadelphia, Pa.).

Schneider, Theodor. Über einen Blichfeldtschen Satz aus der Geometrie der Zahlen. Arch. Math. 2, 349–353 (1950).

Let $\mu=\mu(E)$ be a nonnegative countably additive set function defined for Borel sets E in n-dimensional space R_n. Let $\rho=\liminf_{w\to\infty} w^{-n}\mu(W)$, $\rho'=\limsup_{w\to\infty} w^{-n}\mu(W)$, where W is an arbitrary n-dimensional cube with side w; and suppose that $0<\rho\leqq\rho'<+\infty$. Let $f=f(z)$ be a bounded nonnegative function vanishing outside a bounded set in R_n, the integral $\int fdz$ over R_n being positive. It is shown that, provided certain integrability conditions are satisfied, it is possible, for each $\epsilon>0$, to find points ζ and ζ' such that

$$\int f(z-\zeta)d\mu>\rho\int fdz-\epsilon,$$

$$\int f(z-\zeta')d\mu<\rho'\int fdz+\epsilon.$$

If μ is periodic in each of the coordinates, then $\rho=\rho'$, and the points ζ, ζ' may be chosen to satisfy

$$\int f(z-\zeta)d\mu\geqq\rho\int fdz\geqq\int f(z-\zeta')d\mu.$$

C. A. Rogers (London).

Rogers, C. A. On theorems of Siegel and Hlawka. Ann. of Math. (2) 53, 531–540 (1951).

The main theorem proved is the following. Let K be any convex body in n-dimensional space, symmetrical about the origin O, whose volume does not exceed a certain number depending only on n; let Λ be any lattice of determinant 1. Then there exists a transformation of the special form $x_i=\omega_i x_i'$, $i=1, 2, \cdots, n$, with $\omega_1\cdots\omega_n=1$, such that the transformed lattice Λ' has no point other than O in K. This theorem is a powerful generalization of a result of Siegel [see Davenport, Acta Arith. 2, 262–265 (1937)], Siegel's result being the case when K is a sphere. The whole difficulty of the proof lies in extending Siegel's result from the case of a sphere to that of an arbitrary ellipsoid. The essential lemma (lemma 2) is one which in effect permits the introduction of a single cross-product term into the inequality defining the sphere; once this has been achieved, the further extension to an arbitrary ellipsoid is not difficult.

H. Davenport (London).

Blaney, H. Indefinite ternary quadratic forms. Quart. J. Math., Oxford Ser. (2) 1, 262–269 (1950).

Let $Q(x, y, z)$ be an indefinite ternary quadratic form with determinant $D<0$. Then for any real x_0, y_0, z_0, there exist integers x, y, z such that $0<Q(x+x_0, y+y_0, z+z_0)\leqq(-4D)^{\frac{1}{3}}$. This is true with strict inequality unless $Q(x+x_0, y+y_0, z+z_0)$ is equivalent to a positive multiple of either x^2+yz or $(x-y)(x+y+1)+2z^2$, in which cases it is not.

J. F. Koksma (Amsterdam).

Davenport, H. Note on a binary quartic form. Quart. J. Math., Oxford Ser. (2) 1, 253–261 (1950).

Let x and y be linear forms in two variables u and v with real coefficients and with determinant 1. Then there exist integers u and v, not both zero, such that

$$(1)\qquad |x^2(x^2-y^2)|\leqq 1/(1+2\sqrt{5}).$$

This constant is the least possible, in the sense that there exist special linear forms x, y for which (1) has no solution with strict inequality. *J. F. Koksma* (Amsterdam).

Petersson, Hans. Konstruktion der Modulformen und der zu gewissen Grenzkreisgruppen gehörigen automorphen Formen von positiver reeller Dimension und die vollständige Bestimmung ihrer Fourierkoeffizienten. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1950, 417–494 (1950).

The construction of the modular and automorphic forms of positive dimension is accomplished by making use of their algebraic properties and relationships, and avoids the extensive use of analytical processes found in the investigations of Hardy and Ramanujan, Rademacher and Zuckerman. The present investigation also differs from the others in that no restrictions are placed on the position or order of the poles.

The modular forms of positive dimension are represented as linear combinations of certain basic functions, analogous to the representation of rational functions by partial fractions. If $F(z)$ is a modular form of positive dimension $r-2$ and has a simple pole at $z=\tau_0$ the corresponding basic function $H(\tau_0, z)$ is obtained by interchanging the roles of variable and parameter in a Poincaré series $H(\tau, z)$. As a function of τ, $H(\tau, z)$ is a modular form of negative dimension $-r<-2$ with a simple pole at $\tau=z$. In case of a pole of higher order, derivatives of $H(\tau, z)$ are also used. Corresponding to a pole at $z=\infty$, certain Fourier series F_{r-2} are used. Not all linear combinations of these basic functions are modular, but necessary and sufficient conditions for modularity are found.

The $H(\tau, z)$ and F_{r-2} are related to other Poincaré series G of dimension $-r$ having poles at ∞. The Fourier coefficients of the G are determined and, from these, the Fourier coefficients of $F(z)$ are found. These coefficients of $F(z)$ finally depend on the coefficients of the principal parts of $F(z)$ at each of its poles, the Fourier coefficients of the G and the values of the G and their derivatives at the poles of $F(z)$.

The results for the automorphic forms, with certain restrictions on their transformation groups, are quite similar. A symmetry property of the modular group is obtained and several isolated topics relating to modular forms are discussed in the light of the above results.

H. S. Zuckerman (Seattle, Wash.).

Gut, Max. Euler'sche Zahlen und Klassenanzahl des Körpers der $4\mathfrak{l}$-ten Einheitswurzeln. Comment. Math. Helv. 25, 43–63 (1951).

The Bernoulli numbers B_m are defined by the identity $ve^v(e^v-1)^{-1}=\sum_{m=1}^{\infty}B_m(m!)^{-1}v^m$ and the Euler numbers E_m by the identity $2(e^v+e^{-v})^{-1}=\sum_{m=0}^{\infty}E_m(m!)^{-1}v^m$. Let l be an odd prime; denote by k the field of lth roots of unity. Then Kummer's theorem asserts that the class number of k fails to be divisible by l if and only if the numerators of the first $\frac{1}{2}(l-1)$ Bernouilli numbers of even indices are all relatively prime to l. Let K be the field of 4lth roots of unity, obtained by adjoining $(-1)^{\frac{1}{4}}$ to k. Then the author proves that a necessary and sufficient condition for the class number of K not to be divisible by l is that the numerators of the first $\frac{1}{2}(l-1)$ Bernouilli numbers of even indices and the first $\frac{1}{2}(l-1)$ Euler numbers of even indices are all relatively prime to l. *C. Chevalley* (New York, N. Y.).

Hartman, S. Sur une condition supplémentaire dans les approximations diophantiques. Colloquium Math. 2, 48–51 (1949).

En utilisant la théorie des fractions continues, l'auteur montre le théorème suivant, qui donne la réponse à une question qui lui avait été posée par S. Mazur: Soit $\xi>0$ un nombre irrationnel. Quels que soient les entiers $a\geqq 0$, $b\geqq 0$ et $s>0$, il existe une infinité de couples de nombres naturels u, v, satisfaisant aux conditions

$$(1)\qquad \left|\xi-\frac{u}{v}\right|\leqq\frac{2s^2}{v^2},$$

$$(2)\qquad u\equiv a \pmod{s},\qquad v\equiv b \pmod{s}.$$

L'auteur fait remarquer que l'exposant dans le numérateur du membre droit de (1) ne peut pas être diminué; pour s'en convaincre il suffit de poser $a=b=0$ dans (2). [Le référent remarque qu'on peut diminuer un peu le coefficient 2 dans (1) en utilisant des théorèmes connus sur les approximations diophantiques linéaires non-homogènes, comme il montrera dans une note dans Simon Stevin]. Enfin l'auteur applique son résultat en montrant que

$$\liminf\,(\sin\alpha n)^n=\liminf\,(\cos\alpha n)^n=-1,$$

où α est un nombre réel incommensurable avec π.

J. F. Koksma (Amsterdam).

Jarník, Vojtěch. Sur les approximations diophantiques linéaires non homogènes. Acad. Tchèque Sci. Bull. Int. Cl. Sci. Math. Nat. 47 (1946), 145–160 (1950).

Let $\|x\|$ denote the minimum of $|n+x|$ $(x=0, \pm1, \pm2, \cdots)$ and let Θ_{ij} be real numbers $(1\leqq i\leqq r,\ 1\leqq j\leqq s)$. For real α_i put

$$\psi_1(t;\alpha_1,\cdots,\alpha_r)=\min_{|a_j|\leqq t}\max_i\|\textstyle\sum_j\Theta_{ij}a_j+\alpha_i\|,$$

$$\psi_2(t)=\min_{|b_i|\leqq t}\max_j\|\textstyle\sum_i\Theta_{ij}b_i\|,$$

where a_j, b_i are integers,

$$(a_1,\cdots,a_s)\neq(0,\cdots,0);\quad(b_1,\cdots,b_r)\neq(0,\cdots,0).$$

Put $\psi_1(t)=\psi_1(t;0,\cdots,0)$. The author considers how the behavior of $\psi_1(t;\alpha_1,\cdots,\alpha_r)$ and $\psi_1(t)$ is determined by that of $\psi_2(t)$, the principal results being: (I) $\liminf_{t\to\infty}\psi_2(t)t^{r/s}>0$ implies $\liminf_{t\to\infty}\psi_1(t)t^{s/r}>0$. (This is a particular case of a result of Dyson [Proc. London Math. Soc. (2) 49, 409–420 (1947); these Rev. 9, 271].) (II) Let $\phi(t)$ be a continuous function with $\phi(0)=0$ and $\phi(t)t^{-\eta}$ increasing monotonely to ∞ for some $\eta=0$. Then $\limsup_{t\to\infty}\phi(t)\psi_2(t)>A>0$ implies $\liminf_{t\to\infty}\sup_{0\leqq\alpha_j<1}\rho(t)\psi_1(t;\alpha_1,\cdots,\alpha_r)\leqq\Delta$ where $\rho(t)$ is the inverse function of $\phi(t)$ and Δ is a given function of A, r, s, η only. (III) If $\limsup_{t\to\infty}\phi(t)\psi_2(t)<\infty$ then there is an $(\alpha_1,\cdots,\alpha_r)$ such that $\liminf_{t\to\infty}\rho(t)\psi_1(t;\alpha_1,\cdots,\alpha_r)>0$. (IV) Let $\sigma(t)$ be continuous and monotonely increasing, $\sigma(0)=0$. Suppose that $\sum_{x=1}^{\infty}x^{s-1}/\sigma^r(x)$ converges. Then $\liminf_{t\to\infty}\sigma(t)\psi_1(t;\alpha_1,\cdots,\alpha_r)=\infty$ for almost all $(\alpha_1,\cdots,\alpha_r)$. (V) Suppose that $\sum_{x=1}^{\infty}x^{s-1}/\sigma^r(x)$ diverges and that further $\lim_{t\to\infty}\sigma(t)t^{-s/r}=\infty$, $\liminf_{t\to\infty}t^{r/s}\psi_2(t)>0$. Then

$$\liminf_{t\to\infty}\sigma(t)\psi_1(t;\alpha_1,\cdots,\alpha_r)=0$$

for almost all $(\alpha_1,\cdots,\alpha_r)$. The author has previously obtained somewhat similar results in which the rôles of lim sup and lim inf are interchanged [Časopis Pěst. Mat. Fys. 68, 103–111 (1939)].

J. W. S. Cassels.

Chabauty, Claude, et Lutz, Élisabeth. Approximations diophantiennes linéaires réelles. II. Problème non homogène. C. R. Acad. Sci. Paris 231, 938–939 (1950).

En utilisant les théorèmes de la note précédente [même tome, 887–888 (1950); ces Rev. 12, 483] les auteurs montrent deux théorèmes sur la solution approximative d'un système linéaire non homogène, qui généralisent et précisent des théorèmes connus de Blichfeldt et de Khintchine, à savoir les théorèmes qui sont cités dans le livre du rapporteur [Diophantische Approximationen, Springer, Berlin, 1936, p. 78, théorème 6; p. 82, théorème 3; p. 86, théorème 6]. Comme l'autrice fait remarquer dans un mémoir suivant il faut lire au lemme 3: $\rho_{k+1}\geqq(2q^{\frac12}+1)q^{\frac12}\rho_k$; quatres lignes avant le théorème 1: $\psi(s)\geqq\{24q^2t(s)\}^{-1}$; au théorème 2: $f(p,n)=(p^{\frac12}(p^{\frac12}+1))^{-1}(4n)^{-n/p}\geqq(3p)^{-1}(4n)^{-n/p}$.

J. F. Koksma (Amsterdam).

ANALYSIS

Dresden, Arnold. The Schwarz inequality and the order of operations. Scripta Math. 16, 259–260 (1950).

Let s, i, m denote respectively the operations of squaring, integration over a range R, and multiplying where the operations are performed from right to left. Then the inequality of Schwarz,

$$\left[\int_R fgdR\right]^2\leqq\int_R f^2dR\cdot\int_R g^2dR,$$

is represented symbolically in the form $sim\leqq mis$. The author raises the question about other inequality arrangements. For example is $ims\leqq mis$ or $mis\leqq ims$. Let $f=t$, $g=t^2$, $R=(0,1)$. Then

$$\int_R f^2g^2dR=1/7,\quad \int_R f^2dR\cdot\int_R g^2dR=1/15.$$

Let $f=\sin t$, $g=\cos t$, and let R be the interval $(0,\pi)$.

$$\int_R f^2g^2dR=\pi/8,\quad \int_R f^2dR\cdot\int_R g^2dR=\pi^2/4.$$

These show that neither relation is valid. It is shown in the note that, apart from trivial exceptions, the Schwarz inequality relation is the only one that holds.

R. L. Jeffery (Kingston, Ont.).

Bonsall, F. F. Inequalities with non-conjugate parameters. Quart. J. Math., Oxford Ser. (2) 2, 135–150 (1951).

The author discusses a variety of inequalities, mostly connected with Hilbert's double series theorem [Hardy, Littlewood and Pólya, Inequalities, Cambridge University Press, 1934, chap. 9], and mostly involving parameters $p>1$, $q>1$, with $1/p+1/q>1$ ("nonconjugate parameters"). He applies a new technique which yields much simplified proofs of known results as well as new results. We quote some typical examples. Let $\int f(x)^pdx=F$, $\int g(y)^qdy=G$ (all integrals over $(0,\infty)$), and put $\lambda=1/p'+1/q'<1$. (1) V. Levin [J. London Math. Soc. 11, 119–124 (1936)] gave an explicit value for the constant $K(p,q)$ in Hardy, Littlewood and Pólya's result

$$(*)\qquad \iint f(x)g(y)(x+y)^{-\lambda}dxdy\leqq K(p,q)F^{1/p}G^{1/q}.$$

The author gets the same constant very briefly. (2) A generalization to the nonconjugate case of a theorem with a general kernel in place of $(x+y)^{-\lambda}$ [for the conjugate case see Hardy, Littlewood and Pólya, op. cit., p. 229]. (3) Inequalities involving the integral in (*) and equimeasurable decreasing rearrangements of f and g. (4) If $g(y)$ is nonincreasing and $0<\lambda<1$ then

$$\int\int f(x)g(y)|x-y|^{-\lambda}dxdy \leqq KF^{1/p}G^{1/q}$$

with an explicit K. (5) Sharpened forms of the extension of Hölder's inequality for series to nonconjugate p and q, and a somewhat analogous inequality for integrals. (6) Generalizations to n nonconjugate parameters. (7) A deduction from the integral form of Hilbert's theorem of a sharpened version of the series form [due in a special case to Levin, J. Indian Math. Soc. N.S. 2, 111–115 (1936)] with a larger kernel. (8) A simple proof that the constant $\pi \csc(\pi/p)$ in Hilbert's theorem is the best possible. *R. P. Boas, Jr.*

Ahiezer, N. I. The work of academician S. N. Bernšteĭn on the constructive theory of functions (for his seventieth birthday). Uspehi Matem. Nauk (N.S.) 6, no. 1(41), 3–67 (1951). (Russian)

This is a survey of Bernšteĭn's work on approximation, interpolation, quadrature formulas, inequalities, and analytic functions of a real variable. *R. P. Boas, Jr.*

Avakumović, V. G., and Aljančić, S. The determination of the best bound for the derivative when certain properties of the function and the remaining derivatives are known. Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka 198, 197–210 (1950). (Serbo-Croatian)

See Acad. Serbe Sci. Publ. Inst. Math. 3, 235–249 (1950); these Rev. 12, 484.

van Herk, C. G. G. A class of completely monotonic functions. Compositio Math. 9, 1–79 (1951).

This is a detailed study of functions

$$F(z)=\int_0^1(1-t+tz)^{-1}d\chi(t),$$

where $\chi(t)$ is nondecreasing, and of the moment problem: $x_n \uparrow \infty$, $a_n>0$, $F(x_n)=a_n$. Functions $F(z)$ generalize the Stieltjes transform and have a similar inversion formula. The author finds necessary and sufficient conditions for the existence of a solution of his moment problem, in terms of certain determinants; and also necessary and sufficient conditions for the uniqueness of the solution; a particular sufficient condition is the divergence of $\sum x_n^{-\frac{1}{2}}$. The paper concludes with some examples and applications, and the construction of an $F(x)$ which decreases arbitrarily slowly as $x\to\infty$. *R. P. Boas, Jr.* (Evanston, Ill.).

Siddiqi, Jamil Ahmad. Sur un théorème de M. Mandelbrojt. C. R. Acad. Sci. Paris 232, 2070–2071 (1951).

The author extends a theorem of Mandelbrojt on generalized quasi-analyticity [Ann. Sci. École Norm. Sup. (3) 63, 351–378 (1947); these Rev. 9, 229, 735], imposing conditions which in general require fewer derivatives to vanish in order to make the function in question vanish identically. The detailed statement is too long to reproduce here.

R. P. Boas, Jr. (Evanston, Ill.).

Titus, Charles J. A topological characterization of a class of affine transformations. Duke Math. J. 18, 321–330 (1951).

An oriented closed curve in the plane x, y is said to be of nonnegative circulation if it has a nonnegative order with respect to any point not lying on the curve. C. Loewner has proved the following theorem [Ann. of Math. (2) 49, 316–322 (1948); these Rev. 9, 502]: Given a Lebesgue-summable function $k(t)$ $(0\leqq t\leqq 1)$, in order that any curve which may be represented parametrically in the form

$$(1)\qquad x=x(t),\quad y=-\int_0^1 k(\tau)x(t-\tau)d\tau \qquad (0\leqq t\leqq 1)$$

with a continuous function $x(\tau)$ such that $x(\tau-1)=x(\tau)$ $(-\infty<\tau<\infty)$, it is necessary and sufficient that $k(t)$ be equivalent (in the sense of Lebesgue measure) to a function $k_1(t)$ such that $k_1'(t)=\int_{-\infty}^{\infty}e^{-rt}d\mu(r)$, $\mu(r)$ being a nondecreasing function.

In the present paper, the discrete analog to this theorem is proved. The rôle of $k(t)$ is given to a finite sequence of real numbers $a_1, a_2, \cdots, a_m$, and that of the curve (1) to the oriented closed polygon determined by the sequence of points

$$(x_1, y_1), (x_2, y_1), (x_2, y_2), \cdots, (x_m, y_m), (x_1, y_m), (x_1, y_1),$$

where $y_s=-\sum_{r=1}^m a_r x_{s-r+1}$ $(x_{s-m}=x_s)$. A necessary and sufficient condition that the sequence $\{a_k\}$ generate only polygons of nonnegative circulation is that there exist nonnegative numbers c_r, d_r such that $a_{k+1}-a_k=\sum_{r=1}^q c_r d_r^k$ $(k=1, 2, \cdots, m-1;\ q=[m/2])$. *B. Sz.-Nagy* (Szeged).

Mandelbrojt, S., and Brunk, H. D. A composition theorem for asymptotic series. Duke Math. J. 18, 297–306 (1951).

Les auteurs établissent une propriété des représentations asymptotiques d'un couple de fonctions analytiques, en liaison avec un énoncé donné par l'un d'eux antérieurement [Mandelbrojt, C. R. Acad. Sci. Paris 226, 1155–1157 (1948); ces Rev. 9, 502]. Toutefois, en procédant à partir de ce dernier, on obtiendrait par dualité (transformation de Laplace) une propriété moins générale que celle qui est démontrée directement dans le présent travail. L'énoncé principal est: Si $F(z)$ et $\varphi(z)$ $(z=x+iy)$ sont holomorphes dans le demi-plan $x>0$ et continues sur la frontiere, s'il existe $\delta>0$ tel que $F(z)=O(|z|^\delta)$, $\phi(z)=O(|z|^\delta)$ quand $z\to 0$, et si F et ϕ admettent des représentations asymptotiques $|F-\sum_1^n a_k z^{-k}|<M_n|z|^{-n}$, $|\phi-\sum_1^n b_k z^{-k}|<M_n'|z|^{-n}$ avec $a_{2k-1}=b_{2k-1}=0$, $a_{2k}b_{2k}=0$, $k\geqq 1$, $\sum_1^\infty M_nM_n'/M_{n+1}M'_{n+1}=\infty$, $\log M_n$ et $\log M_n'$ étant fonctions convexes de n, alors, ou bien $F\equiv 0$ ou bien $\phi\equiv 0$. Un lemme d'intérêt général (lemme 2) est donné au cours de la démonstration; celle-ci utilise les propriétés du produit de composition

$$H_c(w)=\int_{\Gamma_c}F_1(s)\phi_1(w-s)ds,$$

Γ_c étant la somme de deux segments $[c-i\pi/2, u-c-i\pi/2]$ et $[u-c+i\pi/2, c+i\pi/2]$, $F_1(s)$ et $\phi_1(s)$ résultant de F et ϕ par $z=e^s$, avec $w=u+iv$. Extension au cas de n fonctions.

P. Lelong (Lille).

van der Waerden, B. L. On the method of saddle points. Appl. Sci. Research B. 2, 33–45 (1951).

Debye's method of saddle points evaluates integrals of the form

$$I=\int_{C'}we^{-\lambda u}dv$$

for large values of λ. In the simplest case λ is real and positive and the real part of u goes to $+\infty$ at both ends of the contour C'. It is assumed here that v and w are analytic functions of u which have only algebraic branch points and are defined on the map C of the contour C' and at all points of the u-plane to the right of C; it is also assumed that in this domain $|wdv/du| \leqq Be^{\epsilon|u|}$ apart from the immediate neighbourhood of a finite number of poles.

The usual procedure is to find first the paths of steepest descent through the branch points and then deform the path of integration C. The author states that "this is completely unnecessary. The only thing we have to do in order to find the asymptotic expansion of an integral I is, to introduce λu or (if λ is real) u as a new variable, to draw the contour C in the u-plane, $\cdots$, to determine the branch points and poles of the functions occurring in the integral, to expand the integrand in a power series in the neighbourhood of every branch point and pole and to integrate term by term."

The investigation arose from the consideration of Sommerfeld's solution of the problem of the propagation of radio waves over a plane earth. Owing to the high conductivity of the earth, the evaluation of a certain complex integral is made difficult because a pole lies very near a saddle point. In the present method the contribution of the pole can be split off immediately. The method is closely related to that of a paper by H. Ott [Ann. Physik (5) **43**, 393–403 (1943); these Rev. **8**, 139] dealing with Sommerfeld's problem; and it is shown that the asymptotic series derived here is not only simpler but that it also gives, with the same number of terms, a better approximation than Ott's series. *E. T. Copson* (St. Andrews).

Mayot, Marcel, et Mineur, Henri. Extension de la méthode d'intégration de Gauss aux fonctions présentant des singularités. C. R. Acad. Sci. Paris **229**, 741–742 (1949).

L'auteur se propose de représenter l'intégrale

$$J=\int_0^1 f(x)dx=\int_0^1 \theta(x)F(x)dx$$

par l'expression $I=\sum_{p=0}^{n} H_p f(x_p)$ ($F(x)$ ne présente pas de singularités dans $(0, 1)$; θ est une fonction positive). Les $2(n+1)$ quantités x_p, H_p sont choisies de telle sorte que $I=J$, si $F(x)$ est une polynome de degré $2n+1$. Pour $\theta=1$ on retrouve la méthode de Gauss. Soit $G_m(x)$, $m=0, 1, \cdots$, une suite de polynomes orthogonaux par rapport à la fonction de base $\theta(x)$ et à l'intervalle $(0, 1)$: $\int_0^1\theta(x)G_n(x)G_m(x)dx=0$, $m\neq n$. Les x_p, $p=0, 1, \cdots, n$ sont les racines de G_{n+1}, $0<x_p\leqq 1$;

$$H_p=[1/G'_{n+1}(x_p)]\int_0^1 \theta(x)[G_{n+1}(x)/(x-x_p)]dx.$$

L'auteur donne sept cas, où les x_p et les H_p ont des expressions simples, par ex.:

$$f(x)=F(x)[x/(1-x)]^{\frac{1}{2}}, \quad G_n(x)=x^{-\frac{1}{2}}T_{2n+1}(x^{\frac{1}{2}})$$

(polynome de Tchebychef), $x_p=\cos^2 \frac{1}{2}[(2p+1)/(2n+3)]\pi$, $H_p=[2\pi/(2n+3)]x_p$. *S. C. van Veen* (Delft).

Jackson, F. H. Basic integration. Quart. J. Math., Oxford Ser. (2) **2**, 1–16 (1951).

From the author's introduction: "In a former number of this journal [Quart. J. Math. (1) **41**, 193–203 (1910)] I gave some examples of an operation which I named 'q-integration'. In this present paper the operation is considered more generally and precisely. The earlier sections of the paper deal with the general theory of basic integration and its relation to Riemann's definition of an integral. The later sections contain examples covering a wider range than those given in the previous paper and are chiefly connected with basic integrals analogous to

$$\int_0^\infty \Phi(-x)x^{s-1}dx=\Gamma(s)\varphi(-s),$$

where $\Phi(x)=\varphi(0)+x\varphi(1)/1!+x^2\varphi(2)/2!+\cdots$. This integral, formally due to Ramanujan, is discussed with precision by G. H. Hardy." The basic integration operator is defined by $(1-q)\mathfrak{D}^{-1}\Phi(x)$, where $\mathfrak{D}\Phi(x)=\{\Phi(x)-\Phi(qx)\}/x$. *N. J. Fine* (Philadelphia, Pa.).

Theory of Sets, Theory of Functions of Real Variables

Sierpiński, W. Sur quelques problèmes concernant la congruence des ensembles de points. Elemente der Math. **5**, 1–4 (1950).

This is a collection of elementary sounding problems about sets in Euclidean spaces of 1, 2, or 3 dimensions which are unsolved, solved for only one of the dimensions, or whose only known solution uses the axiom of choice. Example: Does an arbitrary set E contain a point whose removal yields a set not congruent to E? *J. F. Randolph* (Rochester, N. Y.).

Fodor, G., and Ketskeméty, I. Some theorems on the theory of aggregates. Portugaliae Math. **9**, 145–147 (1950).

Let M be any nonempty set, to each nonempty subset of which has been assigned a representative element. Then if x is the number of elements of M which represent not more than n subsets, the relation $2^x-1\leqq nx$ holds. The number of elements of M representing subsets of at most power n is at most n, where n is a natural number, $\aleph_0$, or c (in the last two cases it being assumed M itself has power c). *R. L. Wilder* (Ann Arbor, Mich.).

Morse, Anthony P. Squares are normal. Fund. Math. **36**, 35–39 (1949).

Two plane sets are "finitely equivalent" if and only if they can be split into sets $a_1, a_2, \cdots, a_m$ and $a_1', a_2', \cdots, a_m'$ in such a way that the corresponding subdivisions are congruent. A plane set S is "paradoxical" if and only if it can be split into two sets each of which is finitely equivalent to S. A plane set which is not paradoxical is "normal." The above title is a theorem (all of whose previous proofs used the axiom of choice) which is a special case of: "Each bounded plane set with inner points is normal", whose proof is given here without employing the axiom of choice either directly or indirectly. *J. F. Randolph* (Rochester, N. Y.).

Segal, I. E. Equivalences of measure spaces. Amer. J. Math. **73**, 275–313 (1951).

This paper deals with properties of measure spaces with special emphasis on the removal of separability assumptions. A measure space is called localizable if the Boolean algebra of measurable sets modulo sets of measure zero is complete as a partially ordered set. Much of the paper is devoted to studying and characterizing localizable spaces. It is necessary to distinguish more carefully between various notions

than in the separable case; thus several notions of isomorphism and equivalence are introduced and their relations studied. As theorem 5.1 seems typical for the results of the paper, we state part of its assertion: For a measure space M any one of the following conditions implies all the others: M is localizable; the Radon-Nikodym theorem holds; the algebra of multiplications by the bounded measurable functions is maximal Abelian in the ring of all bounded operators on $L_2(M)$; the Riesz representation theorem for the linear functionals on L_1 is valid. Among other results which the author also proves under more generality than they were previously treated, is the representability of a measure space as a locally compact topological space (of maximal ideals). *F. I. Mautner* (Baltimore, Md.).

Schaerf, H. M. On the role of an intersection property in measure theory. II. Portugaliae Math. **10**, 1–9 (1951).

[Part I appeared in Portugaliae Math. **8**, 95–102 (1949); these Rev. **12**, 167.] The author specializes the results of part I to the case of measures in groups; he derives again, slightly sharpens, and extends to not necessarily Hausdorff topological groups, the classical theorem on the uniqueness of Haar measure. *P. R. Halmos* (Montevideo).

Mibu, Yoshimichi. A generalization of Haar's measure. Proc. Japan Acad. **22**, no. 8, 246–248 (1946).

Let Ω be a locally compact uniform space acted upon by a transitive group H of homeomorphisms that preserve the uniform structure. The existence of an invariant measure in Ω is asserted. The group H becomes a topological group upon defining the neighborhoods of unity to consist of all $h\epsilon H$ such that $hp\epsilon U_\alpha(p)$ for all p in some compact subset of Ω. In case both Ω and H are locally compact and σ-compact it is asserted that the measure in Ω is unique and that sets of equal measure are decomposition equivalent. Several theorems relating the topological properties of Ω and H are stated. No proofs are given. Reviewer's note: More general existence and uniqueness theorems have been obtained by Loomis [Ann. of Math. (2) **46**, 348–355 (1945); Duke Math. J. **16**, 193–208 (1949); these Rev. **6**, 205; **10**, 600]. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Kappos, Dêmêtrios A. Baire and Borel theory for the Caratheodory "Ortsfunktionen". Bull. Soc. Math. Grèce **25**, 130–152 (1951). (Greek. German summary)

An "Ortsfunktion" is the analog of a point function on a Boolean σ-algebra, B. The author uses previous results [Math. Z. **51**, 616–634 (1949); these Rev. **10**, 437] to connect the functional and algebraic structure: starting with all the "Ortsfunktionen" on B which are step-functions on an arbitrary sub-algebra $R\subset B$, the Baire classes can be characterized by use of a suitable Borel extension of R.
J. Dugundji (Los Angeles, Calif.).

Blackwell, David. The range of certain vector integrals. Proc. Amer. Math. Soc. **2**, 390–395 (1951).

Let $u_1, \cdots, u_n$ be finite signed measures defined on a Boolean σ-algebra of subsets of a set X, and let A be a compact subset of n-dimensional Euclidean space. If R is the set of all points of the form $(\int f_1du_1, \cdots, \int f_ndu_n)$, where $f=(f_1, \cdots, f_n)$ varies over all measurable functions from X to A, then R is closed; if, moreover, the u's are nonatomic, then R is convex. These results are generalizations of those of Liapounoff [Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] **4**, 465–478 (1940); these Rev. **2**, 315]; the author's proofs are ingenious modifications of those used by the reviewer [Bull. Amer. Math. Soc. **54**, 416–421 (1948); these Rev. **9**, 574]. An interesting step in the proof of convexity is the following theorem. If u is a nonatomic measure in X and if $f_1, \cdots, f_n$ are real-valued functions integrable with respect to u, then there exists a Boolean σ-algebra $\mathcal{S}$ of measurable sets such that u is nonatomic on $\mathcal{S}$ and such that $\int_E f_i du = u(E)\int f_i du$, whenever $E\epsilon\mathcal{S}$, $i=1, \cdots, n$. *P. R. Halmos* (Montevideo).

Nöbeling, Georg. Eine Bemerkung über die Länge einer stetigen Kurve. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1949**, 41–45 (1950).

For L any one of a large class of linear measure functions and for B an arc in Euclidean space R_n, it has been proved [Jber. Deutsch. Math. Verein. **52**, 132–160 (1942); these Rev. **5**, 114] that the ordinary length $l(B)=L(B)$. This result is generalized. Let K be the range of a continuous function f from a line segment to R_n. For $t_0<t_1<\cdots<t_k$ a subdivision of the domain segment, join the points $f(t_{k-1})$ and $f(t_k)$ by a segment, sum such lengths, and let $L_f(K)$ be the upper limit of such sums. For $j=1, 2, \cdots, \infty$ let K_j be the set of points of K of multiplicity j. With L a linear measure function as above (and with $\infty\cdot 0=0$ and $\infty\cdot p=\infty$ for $p>0$), the main result of this paper is that $L_f(K)=\sum_{j=1}^{\infty} jL(K_j)$. *J. F. Randolph.*

Kametani, Syunzi, and Enomoto, Shizu. On differentiation of set-functions with some of its applications. Osaka Math. J. **3**, 1–9 (1951).

Ce mémoire s'agit de la dérivation abstraite des fonctions d'ensemble au moyen d'un procédé plus particulier que celui des filtres [le rapporteur, J. Math. Pures Appl. (9) **15**, 391–409 (1936)]. Dans les 3 premiers §, les auteurs reprennent les principaux résultats obtenus dans le mémoire cité, renvoient à ce mémoire pour quelques-uns d'entre eux, et répètent la démonstration de quelques autres. Dans le § 4, ils donnent une démonstration du résultat suivant: Dans un espace métrique localement compact (au sens de Bourbaki), il existe un système dérivant au sens fort [notion définie par le rapporteur, C. R. Acad. Sci. Paris **201**, 579–581 (1935); **224**, 1137–1139 (1947); ces Rev. **8**, 572]. *R. de Possel.*

Sargent, W. L. C. Some properties of C_λ-continuous functions. J. London Math. Soc. **26**, 116–121 (1951).

For $\lambda>0$, $\mu=\max(\lambda-1, 0)$, $f(x)$ is C_λ-continuous in the closed interval $[a, b]$ if $f(x)$ is integrable in the $C_\mu P$ sense [J. C. Burkill, J. London Math. Soc. **11**, 220–226 (1936)] and

$$\lim_{h\to 0} C_\lambda(f, x, x+h) = \lim_{h\to 0} C_\mu P\int_x^{x+h} |x+h-t|^{\lambda-1}f(t) \Big/ \int_x^{x+h} |x+h-t|^{\lambda-1}dt = f(x), \quad a\leqq x\leqq b.$$

The C_λ-derivative, $C_\lambda Df(x)$, is the limit as $h\to 0$ of $(\lambda+1)\{C_\lambda(f, x, x+h)\}/h$. For λ a nonnegative integer, M_λ-continuity is defined in the same way except that $C_{\lambda-1}P$-integrability is replaced by $GM_{\lambda-1}$-integrability [H. W. Ellis, Canadian J. Math. **1**, 113–124 (1949); these Rev. **10**, 520]. Similarly by changing the sense of integrability in the definition of C_λ-derivatives it is possible to define M_λ-derivatives. Using methods analogous to those of Tolstoff [Rec. Math. [Mat. Sbornik] N.S. **5**(47), 637–645 (1939); these Rev. **1**, 206] for approximate continuity and derivatives, the author obtains relations between C_λ-continuity and derivatives and ordinary continuity enabling

her to establish the following mean value theorem. If $f(x)$ is C_λ-continuous for $a \leqq x \leqq b$, while $C_\lambda Df(x)$ exists (finite or infinite) for $a<x<b$, then there is a point c such that $[f(b)-f(a)]/(b-a) = C_\lambda Df(c)$. This theorem was previously obtained by the author in the case $\lambda=1$ [Proc. London Math. Soc. (2) 40, 235–254 (1935), p. 239]. Another result obtained is that a C_λ-continuous function takes all values between any two values. The paper concludes with a proof that the class of functions M_n-continuous on an interval coincides with the class of functions C_n-continuous on the interval so that the results obtained for C_λ-continuous functions hold also for M_λ-continuous functions. The second result then gives a generalization of a theorem of Ellis [op. cit., p. 124]. *R. L. Jeffery* (Kingston, Ont.).

Sargent, W. L. C. On generalized derivatives and Cesàro-Denjoy integrals. Proc. London Math. Soc. (2) 52, 365–376 (1951).

The author [Proc. London Math. Soc. (2) 47, 212–247 (1941); these Rev. 3, 228] has given a descriptive definition of the C_nD integral (n a nonnegative integer) which is equivalent to Burkill's C_nP integral [ibid. (2) 39, 541–552 (1935)]. The definition is by induction using Cesàro derivatives. The present paper gives a straightforward definition of an integral, the V_nD integral which is equivalent to the C_nD integral but based on properties of the generalized derivatives of de La Vallée Poussin. If $F(x)$, defined and finite in some neighborhood of a point t, is such that

$$F(x)=\sum_{r=0}^{n} a_r(x-t)^r/r!+o((x-t)^n)$$

as $x\to t$ where $a_0=F(t)$ and $a_1, a_2, \cdots, a_n$ are finite and independent of x, then a_n is the nth generalized derivative of $F(x)$ at the point t and is denoted by $F_n(t)$. If $F_n(x)$ exists at all points of an interval the function $\epsilon_n(t, x)$ is defined by the equation

$$F(x)=\sum_{r=0}^{n} \frac{(x-t)^r}{r!} F_r(t)+\frac{(x-t)^n}{n!}\epsilon_n(t, x)$$

whenever t and x lie in the interval, $t\neq x$; $\epsilon_n(t, t)=0$.

The finite function $f(x)$ is V_n-AC^* over a bounded set E if (i) there exists $F(x)$ having an nth generalized derivative $F_n(x)$ which coincides with $f(x)$ in an interval containing E and $F(x)$ is such that (ii) to each positive ϵ corresponds a δ such that, where

$$\omega_n(c, d)=\max\{\overline{\text{bound}}_{c<x<d}|\epsilon_n(c, x)|, \overline{\text{bound}}_{c<x<d}|\epsilon_n(d, x)|\},$$

$$\sum_{r=1}^{m}\omega_n(a_r, b_r)<\epsilon$$

for all finite sets of nonoverlapping intervals $(a_1, b_1), \cdots, (a_m, b_m)$ with end points on E and such that $\sum_{r=1}^{m}(b_r-a_r)<\delta$. In Theorem IV condition (ii) is shown to be independent of the particular $F(x)$ chosen. If $[a, b]$ is the sum of an enumerable number of closed sets over which $f(x)$ is V_n-AC^* then $f(x)$ is V_n-ACG^* over $[a, b]$. If $f(x)$ coincides with $F_n(x)$ in some neighborhood of a point t and if $F_{n+1}(t)$ exists then $f(x)$ is V_n-differentiable at t with V_n-derivative $V_nDf(t)$ equal to $F_{n+1}(t)$.

Definition of the V_nD integral. The function $\theta(x)$ is integrable in the V_nD sense in (a, b) if there is a function $f(x)$ which is V_n-ACG^* over $[a, b]$ and such that $V_nDf(x)=\theta(x)$ almost everywhere in (a, b). When these conditions are satisfied $f(b)-f(a)$ is called the definite V_nD integral of $\theta(x)$ in (a, b). The paper concludes with a proof that the V_nD integral is equivalent to the C_nD integral and therefore also to the C_nP integral. *R. L. Jeffery.*

Carr, R. E., and Hill, J. D. Pattern integration. Proc. Amer. Math. Soc. 2, 242–245 (1951).

Soit $P=\{\alpha_p{}^n\}$ une suite prenant les valeurs 0 et 1, définie pour $0<p\leqq n$, $f(x)$ une fonction intégrable au sens de Riemann pour $0\leqq x\leqq 1$, et $\xi_k{}^n$ des nombres tels que $k/n\leqq \xi_k{}^n \leqq (k+1)/n$. On pose $S_n=n^{-1}\sum_{k=1}^{n}\alpha_k{}^n f(\xi_k{}^n)$. Les auteurs recherchent des hypothèses simples sur P pour que S_n ait une limite, qu'ils nomment alors "special pattern integral." Ils énoncent essentiellement le résultat que voici: Supposons que P vérifie la condition suivante: Quelle que soit la suite $p_n\leqq n$ telle que p_n/n ne tende pas vers zéro, $p_n{}^{-1}\sum_{k=1}^{p_n}\alpha_k{}^n$ a une même limite α. Alors S_n a une limite égale à $\alpha\int_0^1 f(x)dx$. La démonstration n'est donnée que dans le cas où $\alpha_p{}^n$ ne dépend pas de n: "fixed pattern". L'exemple suivant est indiqué: Soit q un entier fixe, l_n un entier indépendant de n, posons $\alpha_p{}^n=1$ si $p\equiv l_n \pmod q$ et $\alpha_p{}^n=0$ dans le cas contraire. Alors α existe et est égal à q^{-1}.

R. de Possel (Alger).

Yano, Shigeki. Notes on Fourier analysis (XIX): A remark on Riemann sums. Tôhoku Math. J. (2) 2, 1–3 (1950).

Let $f(x)\epsilon L$ have period 1 and let $f_n(x)=n^{-1}\sum_{k=1}^{n}f(x+k/n)$. The object of this paper is to prove that: (1) If $f\epsilon\text{Lip}(\alpha, p)$, $0<\alpha\leqq 1$, $p\geqq 1$, and if $\sum_1^\infty n_k{}^{-\alpha}<\infty$, then $f_{n_k}(x)\to\int_0^1 fdx$ almost everywhere; (2) If $\int_0^1|f(x+t)-f(x)|dx=O(\log^{-s}t^{-1})$, $s>1$, and if $\sum \log^{-s} n_k<\infty$, then $f_{n_k}(x)\to\int_0^1 fdx$ almost everywhere. [See also the reviewer, Mat. Tidsskr. B. 1948, 60–62; these Rev. 10, 360.] *R. Salem.*

Łoś, Jerzy. Un théorème sur les superpositions des fonctions définies dans les ensembles arbitraires. Fund. Math. 37, 84–86 (1950).

Soit E un ensemble quelconque; l'auteur considère des fonctions de n variables x_i, définies pour $x_i\epsilon E$, et prenant leurs valeurs dans E. Il démontre que, pour toute suite infinie de telles fonctions (pour lesquelles n prend des valeurs quelconques), il existe une fonction φ de deux variables telle que toute fonction de la suite soit une superposition finie de la fonction φ. La démonstration est très rapide et s'appuie sur des propriétés analogues, mais plus particulières, dues à Sierpiński et Webb. *R. de Possel.*

Sierpiński, Wacław. Sur une suite infinie de fonctions continues dont toute fonction d'accumulation est non mesurable. Acad. Serbe Sci. Publ. Inst. Math. 1, 5–10 (1947).

A function f is called an accumulation function of a sequence f_k, $k=1, 2, \cdots$, if for each $\epsilon>0$ and each finite system $x_1, x_2, \cdots, x_m$, there are an infinity of indices such that $|f_k(x_i)-f(x_i)|<\epsilon$, $i=1, 2, \cdots, m$. The title states the results. There are some minor misprints.

J. F. Randolph (Rochester, N. Y.).

Cafiero, Federico. Criteri di compattezza per le successioni di funzioni generalmente a variazione limitata. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 450–457 (1950).

This note continues the investigations reported in part I [ibid. 8, 305–311 (1950); these Rev. 12, 247]. A sequence $\{f_n(x, y)\}$ defined in D: $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, measurable in y for almost all x and continuous for almost all y, is called

quasiuniformly convergent semi-regularly with respect to x if for each $\epsilon>0$ there is a set I_ϵ of y in $[0, 1]$ with measure less than ϵ such that if we remove from D the points (x, y) with y in I_ϵ the sequence converges uniformly on the remainder of D. The principal theorem is that if $\{f_n(x, y)\}$ is a sequence of functions uniformly bounded on D and generally of bounded variation [cf. the cited reference] and absolutely continuous in y for almost all x in $[0, 1]$, and if furthermore there exist positive α, k such that for all n: $\int_0^1 V_x^{(n)}(y)dy<k$, $\int\int_D|\partial f_n/\partial y|^{1+\alpha}dxdy<k$, then the sequence contains a subsequence which is quasi-uniformly convergent semi-regularly with respect to y. It is also shown that if $\{f_n(x, y)\}$ is a sequence of measurable functions generally of bounded variation on D, and the integrals $\int_0^1 V_x^{(n)}(y)dy+\int_0^1 V_y^{(n)}(x)dx$ are $\leqq$ some constant k, then under rotation to new axes X and Y the analogous condition holds on every rectangle R: $a\leqq X\leqq b$, $c\leqq Y\leqq d$ interior to D. *E. J. McShane.*

Kneser, Martin. Abhängigkeit von Funktionen. Math. Z. **54**, 34–51 (1951).

The author has rediscovered, and proved by different methods, results on functional dependence essentially obtained by Sard [Bull. Amer. Math. Soc. **48**, 883–890 (1942); these Rev. **4**, 153] who stated his results in terms of the m-dimensional measure of a map $y_i=f_i(x_1, \cdots, x_n)$, $i=1, \cdots, m$, and did not explicitly mention the words "functional dependence", but the results on functional dependence follow immediately from Sard's results and minor results of Knopp and Schmidt [Math. Z. **25**, 373–381 (1926)] or, in another form, of the reviewer [Trans. Amer. Math. Soc. **38**, 379–394 (1935)]. The author's treatment is based in a general way on that in the reviewer's paper, which gave results on functional dependence under stronger differentiability hypotheses. The improvement is obtained by use of results of H. Whitney, and extensions of those results, involving differentiability of functions defined on arbitrary subsets of Euclidean spaces. Given m real functions $f_1, \cdots, f_m$, of class C^k, of n real variables, defined on a set A of E_n, and an integer r such that the rank of the Jacobian matrix is at most r at each point of A. The author proves the following theorems. (1) If A is compact and $k\geqq(n-r)/(m-r)$ and $k\geqq1$, then the image of A in E_m under the map $y_i=f_i(x_1, \cdots, x_n)$, $i=1, \cdots, m$, is nowhere dense. This theorem is covered by Sard's theorems 4.1 and 7.1. While Sard has A as an open set, a theorem of Whitney shows that Sard's results imply the author's results. The conclusion of (1) could have included the result that the f's are functionally dependent. An immediate corollary is (2) if $k=\max(n-m+1, 1)$, and $r=m-1$, then $f_1, f_2, \cdots, f_m$ are functionally dependent. This is essentially covered by Sard's theorems 4.1 and 7.2. Another simple corollary is (3) if $k\geqq\max(\frac{1}{2}(n-m+2), 1)$ and $r=m-1$, then $f_1, \cdots, f_m$ are functionally dependent. (3) was not given by Sard.
A. B. Brown (Flushing, N. Y.).

Theory of Functions of Complex Variables

Kestelman, H. Automorphisms of the field of complex numbers. Proc. London Math. Soc. (2) **53**, 1–12 (1951).

A function $\phi(z)$, defined for all complex z, is said to define an automorphism of the field Z of complex numbers if (1) $\phi(z_1+z_2)=\phi(z_1)+\phi(z_2)$, (2) $\phi(z_1z_2)=\phi(z_1)\phi(z_2)$, (3) for every $\zeta\epsilon Z$ the equation $\phi(z)=\zeta$ has a solution. Trivial automorphisms are given by $\phi(z)=z$ and $\phi(z)=\bar{z}$. C. Segre [Atti Accad. Torino **25**, 276–301, 430–457 (1890)] raised the question of the existence of nontrivial ϕ. The author states the history of the problem, gives a new proof that nontrivial ϕ exist, and deduces properties of such ϕ, pointing to their "wild" character. For example, let a Segre function be a function ϕ that satisfies (1) and (2) for all z_1 and z_2 in some field K. If $\phi(z)$ is a nontrivial Segre function in Z and S is any plane set whose interior Lebesgue measure is positive, then $\phi(S)$ is everywhere dense in Z; and if E is a set with both interior and exterior points then $\phi^{-1}(E)$ is of zero interior measure and is nonmeasurable (and the same is true of $\phi(E)$ if ϕ is an automorphism). *I. M. Sheffer.*

Andersson, Bengt. On an inequality concerning the integrals of moduli of regular analytic functions. Ark. Mat. **1**, 367–373 (1951).

Let $f(z)$ be an analytic function, regular in a convex region D and on its boundary C. Let L be a rectifiable curve in D. The author proves that $\int_L|f(z)dz|\leqq(A/2\pi)\int_C V(t)|f(t)dt|$ where $A=4$ and $V(t)$ is the upper limit of the sum of the angles at which the elements of L are seen from the point t on C. If the inner curve L is convex, then $V(t)\leqq2\pi$ and $\int_L|f(z)dz|\leqq A\int_C|f(t)dt|$, $A=4$. The value $A=4$ is not the best possible and the author obtains the refinement $A=3.6$.
G. Springer (Evanston, Ill.).

Eggleston, H. G. The coefficient theory of functions with singularities of the form

$$\left(\frac{1}{c-z}\right)^\sigma\left(\log\frac{1}{c-z}\right)^k\int_{-\infty}^{+\infty}\frac{d\beta(t)}{(c-z)^{it}}.$$

Proc. London Math. Soc. (2) **53**, 476–492 (1951).

R. Jungen's approach to the art of estimating the Taylor coefficients of functions with algebraic-logarithmic singularities [Comment. Math. Helv. **3**, 266–306 (1931)], repeatedly generalized and refined by R. Wilson and by the author, is now developed further so as to apply to functions $f(z)$ of the class indicated in the title. The constant σ is assumed to be real; k is a positive integer or zero; $\beta(t)$ is a function whose total variation on the real axis is finite and whose total variations in the intervals (x, ∞) and $(-\infty, -x)$ tend to zero with prescribed rapidity as x becomes large (the details of the latter requirement are expressed in terms of arg c). The singularity of $f(z)$ at c is algebraic-logarithmic provided $\beta(t)$ is a step function or the integral is replaced by a constant. The author shows that $f(z)=\sum a_nz^n$, where $a_n=\int_{-\infty}^{\infty}a_n(s)d\beta(t)$; here $s=\sigma+it$, and $a_n(s)$ denotes the coefficient of z^n in the expansion of $(c-z)^{-s}[\log 1/(c-z)]^k$. He deduces that if $\beta(t)$ is continuous at $t=0$,

$$a_nc^nn^{1-\sigma}(\log n)^{-k}=\int_{-\infty}^{\infty}n^{it}d\gamma(t)+o(1), \tag{1}$$

where $\gamma(r)=\int_{-\infty}^{r}\{c^{-s}/\Gamma(s)\}d\beta(t)$. If further $\beta(t)$ has a finite discontinuity at $t_0\neq0$, then for any positive null-sequence $\{\epsilon_n\}$ the integers n for which the left member of (1) is in absolute value less than ϵ_n form a sequence of zero density.

If $\phi(z)$ is regular at c and $\phi(c)\neq0$, the singularity at c of $f(z)\phi(z)$ is said to be of weight $[\sigma, k]$; it is of type A provided $\beta(t)=\lambda(t)+\mu(t)$, where $\lambda(t)$ is absolutely continuous and $\mu(t)$ is a jump function; a singularity of type A is of type $A(r)$ provided the corresponding function $\mu(t)$ has precisely r jumps. Weights are ordered lexicographically. If the series $\sum a_nz^n$ has on its circle of convergence singularities of type A only, of which p are of the greatest weight $[\sigma, k]$,

and if one of these p singularities is of type $A(1)$, then for all sufficiently large values of n

$$K_1<|c|^n n^{1-r}(\log n)^{-k}\{|a_{n-1}|+|a_{n-2}|+\cdots+|a_{n-p}|\}<K_2,$$

where K_1 and K_2 are positive constants. If the series $\sum a_n z^n$ has gaps of unbounded length, it has on its circle of convergence a singularity which is not of type $A(r)$, $1\leqq r<\infty$.
G. Piranian (Ann Arbor, Mich.).

Teghem, J. Addenda à la note: "Sur les conditions d'applicabilité d'une méthode de prolongement analytique de Borel." Acad. Roy. Belgique. Bull. Cl. Sci. (5) 37, 20 (1951).

See the same Bull. Cl. Sci. (5) 35, 177–185 (1949); these Rev. 11, 21.

Tomić, M. Sur certaines propriétés des séries de Taylor dont les coefficients sont convexes ou satisfont à d'autres conditions analogues. Acad. Serbe Sci. Publ. Inst. Math. 3, 243–258 (1950).

The author gives a very simple, geometric proof of the following theorem due to Fejér and Szegö [Prace Mat.-Fiz. 44, 15–25 (1937)]: Let a_k, $k=0, 1, 2, \cdots$, be a convex sequence of real numbers and $f(z)=\sum_0^\infty a_k z^k$ be convergent for $|z|<1$. Let further $R_n(z)=\sum_n^\infty a_k z^k$. Then $|R_n(z)|\geqq|R_{n+1}(z)|$ for $n=0, 1, 2, \cdots$ and $|z|\leqq 1$, $z\neq 1$. His proof involves the construction of a polygonal line representing the sum $s_n(z)=\sum_0^n c_k e^{ik\theta}$, $c_k=a_k|z|^k$, the circle K_n through points s_n and s_{n+1} with center w_n such that angle $s_n w_n s_{n+1}$ equals θ and the circle K_n' through w_n and w_{n+1} with center at w_n' such that angle $w_n w_n' w_{n+1}$ equals θ. Due to the convexity of the sequence c_k, point $f(z)$ is found to lie in all the circles K_n and K_n' and hence to be farther from point s_n than from s_{n+1}. By a similar method the author proves that if $a_n\leqq a_{n+1}$ and if for some m the differences $d_{mn}=\sum_0^m(-1)^k[m!/(k!(m-k)!)]a_{n-k}$, $n=0, 1, 2, \cdots$, form a decreasing sequence, then $|s_n(z)|\leqq Ha_n|f(z)|$ for all $n=0, 1, 2, \cdots$ and $|z|\leqq 1$, $z\neq 1$, H being a constant independent of z and n. A similar method likewise gives some information concerning the asymptotic behavior of the polynomial $P_n(\cos\theta)$ occurring in the product expansion $f(re^{i\theta})f(re^{-i\theta})=\sum_0^\infty P_n(\cos\theta)r^n$ when certain assumptions are made concerning the a_k.
M. Marden.

Rauch, Louis M. Some general inversion formulae for analytic functions. Duke Math. J. 18, 131–146 (1951).

Referring to the Bürmann-Lagrange and to the Ward formula, the author deduces, with two proofs, a new representation for the coefficients $b_r=g^{(r)}(w_0)/r!$ of the power series $z=g(w)=z_0+\sum_1^\infty b_r(w-w_0)^r$, the inverse function of $w=f(z)$ $(f'(z_0)\neq 0,\ w_0=f(z_0))$ in the neighbourhood of w_0. Let $\alpha_1, \alpha_2, \cdots, \alpha_r$ be nonnegative integers such that $\alpha_1+\cdots+\alpha_r=r-1$, $\alpha_1+2\alpha_2+\cdots+r\alpha_r=2r-2$, and $S(r)$ the set of all these r-tuples, r fixed. Then $g'(w)=1/f'(w)$,

$$g^{(r)}(w)=\sum_{(\alpha_1,\cdots,\alpha_r)\subset S(r)}(-1)^{r+\alpha_1-1}(2r-\alpha_1-2)!\times(f'(z))^{\alpha_1-2r+1}\prod_{k=2}^{r}\frac{(f^{(k)}(z))^{\alpha_k}}{(\alpha_k)!(k!)^{\alpha_k}}\quad(r\geqq 2).$$

Tables of $S(r)$ are given for values of r from 1 to 8. The result is checked by means of $w=\sin z$. It is applied then to the quintic equation $z^5+\alpha_1 z+\alpha_0=0$, and, finally, to the problem: Given the modulus k of the Jacobian elliptic function, to find q, the parameter of the theta functions, by representing q as a power series in terms of $\epsilon=\frac{1}{2}\{1-(1-k^2)^{\frac{1}{2}}\}/\{1+(1-k^2)^{\frac{1}{2}}\}$. The first four terms were given by Weierstrass. At present, fourteen terms are known: see Lowan, Blanch and Horenstein [Bull. Amer. Math. Soc. 48, 737–738 (1942); these Rev. 4, 90]. They could be checked, as the author states, by his method.
H. Kober.

Walsh, J. L., and Russel, H. G. On simultaneous interpolation and approximation by functions analytic in a given region. Trans. Amer. Math. Soc. 69, 416–439 (1950).

Let $\bar{S}$ be a closed point set interior to the (open) region R. Walsh and Nilson [same Trans. 55, 53–67 (1944); 65, 239–258 (1949); these Rev. 5, 115; 10, 524] have studied approximation on $\bar{S}$ to a function $F(z)$ analytic on $\bar{S}$ but not throughout R. Here simultaneous interpolation and approximation is treated, where there is no necessary connection between $F(z)$ and the values interpolated to and where in some problems $F(z)$ is not required to be analytic. Let C_1 or C_0, respectively, be the boundary of R or S, and consist of a finite number of mutually disjoint Jordan curves; $C_0=C_0^{(1)}+\cdots+C_0^{(s)}$, $\bar{S}=S+C_0$; let points z_k in S and values A_k be given, let $f_M(z)$ be analytic in R, of modulus at most M, $f_M(z_k)=A_k$ $(k=1, \cdots, n)$. The authors deal with 4 problems, taking the measure of approximation of $f_M(z)$ to $F(z)$ as

$$\text{(1)}\qquad \sum_{j=1}^{s}[\max|f_M(z)-F(z)|,\ z \text{ on } C_0^{(j)}];$$

$$\text{(2)}\qquad \int_{C_0}|f_M-F|^2|dz|;$$

$$\text{(3)}\qquad \iint_S|f_M-F|^2dS;$$

$$\text{(4)}\qquad \int_{C_0}|f_M-F|^p|dz|\qquad(p>0),$$

respectively, for $F(z)$ meromorphic in S; $F(z)\epsilon L^2$ on C_0; $F(z)\epsilon L^2$ on S; or $F(z)\epsilon H_p$ in C_0, where H_p is the known Riesz class. They introduce a unique $f(z)$, analytic in C_0, satisfying some condition concerning the boundary and the interpolation condition, and such that its approximation to $F(z)$ is best, in the measure concerned, with a slight alteration in case (1). They show that $f_M(z)\to f(z)$ $(M\to\infty)$ in S, uniformly on any closed subset of S. Moreover, if $\phi(z)$ is harmonic in $R-\bar{S}$, continuous on $\bar{R}-S$, $=0$, or $=1$ on C_0 or C_1, respectively, C_ν is the locus $\phi(z)=\nu$ $(0\leqq\nu\leqq 1)$, R_ν the region bounded by C_ν, then

$$\limsup_{M\to\infty}[\max|f-f_M|,\ z \text{ in } \bar{R}_\nu]^{1/\log M}=e^{(\nu-\rho)/(1-\rho)}\qquad(0\leqq\nu<\rho),$$

$$\limsup_{M\to\infty}[\sup|f_M(z)|,\ z \text{ in } R_\nu]^{1/\log M}=e^{(\nu-\rho)/(1-\rho)}\qquad(\rho\leqq\nu<1),$$

provided that $f(z)$ is analytic in R_ρ $(0<\rho<1)$ but not throughout any $R_{\rho'}$ $(\rho'>\rho)$. The results for problem (4) are based on the theorems 6.2 and 6.1, due to A. Spitzbart and to the reviewer [Bull. Amer. Math. Soc. 52, 338–346 (1946); 49, 437–443 (1943); these Rev. 7, 425; 4, 242], respectively.
H. Kober (Birmingham).

Martin, Yves. Sur quelques séries d'interpolation et de facultés. Bull. Sci. Math. (2) 75, 21–32 (1951).

The author continues his study [cf. Ann. Sci. École Norm. Sup. (3) 66, 311–366 (1949); these Rev. 11, 344] of the convergence of the series $\sum_1^\infty a_n P_n(z)$, $P_n(z)=\prod_1^n(1-z/\lambda_k)$, where the λ_k are now permitted to be complex. If $\sum_1^\infty 1/|\lambda_n|$ converges and if the given series converges for $z=z_0\bar{\epsilon}\{\lambda_n\}$,

then the series converges everywhere in the complex plane, uniformly on any bounded set. The possibility of representing an entire function by a series of this type is discussed. If $\sum_1^\infty 1/|\lambda_n|$ diverges, if $\lambda_n \uparrow$, and if $\arg \lambda_n \to 0$, then the given series converges in a half-plane $\Re z > \Re z_0$ if it converges for $z = z_0 \bar{\epsilon} \{\lambda_n\}$. Here the convergence and absolute convergence are related through the density of the sequence $\{|\lambda_n|\}$. The convergence of series of the type $\sum_1^\infty a_n R_n(z)$, $1/R_n(z) = \prod_1^n (1+z/\lambda_k)$ is also investigated and similar results obtained. For example, if $\sum_1^\infty 1/|\lambda_n|$ diverges and $\arg \lambda_n \to 0$, then the series $\sum_1^\infty a_n P_n(z)$ and $\sum_1^\infty a_n R_n(z)$ have the same abscissa of convergence. Finally, the author considers briefly series of interpolation $\sum_1^\infty a_n Q_n(z)$, $Q_n(z) = \prod_1^n (z-\lambda_k)$, where $\lambda_n \to 0$. *E. N. Nilson.*

Macintyre, Sheila Scott. Overconvergence properties of some interpolation series. Quart. J. Math., Oxford Ser. (2) 2, 109–120 (1951).

A Lidstone series, which has the form

$$\sum [f^{(2i)}(0)\alpha_i(z) + f^{(2i)}(1)\beta_i(z)],$$

where $\alpha_i(z)$ and $\beta_i(z)$ are certain polynomials, if it converges at any nonintegral point, can only represent an entire function. The author shows, however, that a sequence of partial sums of the series will converge to a nonentire function whose derivatives satisfy suitable growth properties, for example

$$(1) \qquad \liminf_{n\to\infty} \max_{z\in\Gamma} |f^{(2n)}(z)| \pi^{-2n} = 0,$$

where Γ is a rectifiable curve joining 0 and 1 on which $f(z)$ is analytic; or to an entire function of what might naturally be called lower exponential type less than π. An example of a nonentire function satisfying (1) is constructed. Similar results are given for the m-point Gontcharoff series [Comm. Inst. Sci. Math. Méc. Univ. Kharkoff [Zapiski Inst. Mat. Mech.] (4) 5, 67–85 (1932)],

$$\sum_{j=0}^{\infty} \sum_{i=1}^{m} P_{mj+i-1}(z) f^{(mj+i-1)}(a_i),$$

where $P_k(z)$ are the Gontcharoff polynomials associated with the sequence $\{a_k\}$, $a_{m+p} = a_p$. *R. P. Boas, Jr.*

Lohin, I. F. The interpolation of functions regular in a half-plane. Doklady Akad. Nauk SSSR (N.S.) 78, 9–12 (1951). (Russian)

A proof is given of the following result: Let the sequence of numbers A_n be given. In order that there exist a function $f(z)$ such that $f(\lambda_n) = A_n$ $(n = 1, 2, \cdots)$, where i) $f(z)$ is regular in the half-plane $\Re(z) > 0$,

$$\text{ii)} \qquad 0 < \lambda_1 < \lambda_2 < \cdots, \quad \lim_{n\to\infty} n/\lambda_n = \sigma < \infty,$$

iii) for arbitrary z in $\Re(z) > 0$, $|f(z)| < e^{k|z|}$, iv) for arbitrary x in $\lambda_1 - \delta \geqq x \geqq \delta > 0$, $|f(x+iy)| < c(\delta)e^{\theta|y|}$, $\theta < \pi\sigma$, it is necessary and sufficient that a) the sequence of polynomials (for some increasing sequence of integers m_k)

$$P_k(t) = \sum_{n=1}^{m_k} A_n e^{-\lambda_n t}/\varphi'(\lambda_n) \quad (\varphi(t) = \prod_{n=1}^{\infty} (1 - t^2/\lambda_n^2))$$

converges in a half-plane $\Re(t) > \alpha$ to a function $F(t)$ that is regular there, b) in some strip $|\Im(t)| \leqq \eta_0$ $(\eta_0 > 0)$, $F(t)$ satisfies

$$|F(t)| = |F(\xi+i\eta)| < \begin{cases} e^{-(\lambda_1-\delta)\xi} & \text{for} \quad \xi > \xi_0(\delta) > 0 \\ e^{-k\xi} & \text{for} \quad \xi < -\xi_0(\delta) \end{cases} \quad (|\eta| \leqq \eta_0).$$

I. M. Sheffer (State College, Pa.).

Nassif, M. On the effectiveness at the origin of product and inverse sets of polynomials. J. London Math. Soc. 26, 232–238 (1951).

The author generalizes results of M. T. Eweida [Proc. London Math. Soc. (2) 51, 81–89 (1949); these Rev. 10, 444] by replacing the condition that $q_n(z)$ has leading coefficient unity by the condition that the upper and lower limits of $R^{-1}[B_n(R)]^{1/n}$, $n\to\infty$, $0 < R \leqq h$, are respectively finite and nonzero, where $B_n(R) = \max |q_n(z)|$ for $|z| = R$. *R. P. Boas, Jr.* (Evanston, Ill.).

Mergelyan, S. N. On a theorem of M. A. Lavrent'ev. Doklady Akad. Nauk SSSR (N.S.) 77, 565–568 (1951). (Russian)

The author gives a new, short and remarkably simple proof of Lavrent'ev's approximation theorem [Sur les fonctions d'une variable complexe representables par des séries de polynômes, Actualités Sci. Ind., no. 441, Hermann, Paris, 1936]: If C is a nowhere dense continuum in the complex z-plane which does not separate the plane, then every continuous function defined on C can be expanded in a uniformly convergent series of polynomials in z. *L. Bers* (Los Angeles, Calif.).

Bernšteĭn, S. N. On weight functions. Doklady Akad. Nauk SSSR (N.S.) 77, 549–552 (1951). (Russian)

Let $\Phi(x)$ be positive on the real axis; the author calls $\Phi(x)$ a weight function $(\Phi(x)\epsilon W)$ if for any continuous $f(x)$ such that $f(x)/\Phi(x) \to 0$ $(|x|\to\infty)$ and any $\epsilon > 0$ there is a polynomial $P(x)$ such that $|f(x) - P(x)| < \epsilon\Phi(x)$. He has shown [Leçons sur les propriétés extrémales . . . , Gauthier-Villars, Paris, 1926] that an even entire function $E(x)$ with nonnegative Maclaurin coefficients is in W if and only if it is of genus 0. In the present paper he proves the following results. Theorem 1. If $\lambda_n = \min |x|^{-1}\Phi(x)^{1/n}$ and $S_0 = \sum \lambda_n$ diverges then $\Phi\epsilon W$. Corollary 1. If Φ is a majorant of finite or quasifinite growth [Bernšteĭn, same Doklady (N.S.) 65, 117–120 (1949); these Rev. 11, 23] then Φ non-ϵW. Theorem 2. Let $\Phi(x) = e^{p(x)}$ be a positive entire function; if $\int_1^R x^{-2}p(x)dx$ is bounded in R then S_0 converges; hence Φ non-ϵW [cf. Ahiezer and Babenko, same Doklady (N.S.) 57, 315–318 (1947); these Rev. 9, 141]. Now let $\Phi\epsilon N$ ("of normal increase") if $\Phi(x) = e^{p(x)} > 0$, $\Phi(x) \uparrow \infty$, $xp'(x) \uparrow \infty$. Then Theorem 2 subsists if $\Phi\epsilon N$. Corollary 2. If the even function $\Phi\epsilon N$ then S_0 converges or diverges with $\int^\infty x^{-2}p(x)dx$. Theorem 3. An entire function $E(x)$ as specified above is of genus 0 if and only if S_0 converges. Corollary 3. If $|G(x)|\epsilon N$ is the absolute value of an entire function of finite degree (exponential type) then it belongs to W or is a majorant of finite growth according as S_0 diverges or converges. *R. P. Boas, Jr.* (Evanston, Ill.).

Mandelbrojt, S., and Ulrich, F. E. Regions of flatness for analytic functions and their derivatives. Duke Math. J. 18, 549–556 (1951).

J. M. Whittaker [Proc. Edinburgh Math. Soc. (2) 2, 111–128 (1930)] a introduit dans la théorie des fonctions entières d'ordre fini la notion de "flat" regions, régions dans chacune desquelles la variation du module de la fonction est relativement faible; cette notion a été étendue et étudiée par J. M. Whittaker [Quart. J. Math., Oxford Ser. (1) 2, 252–258 (1931); Proc. London Math. Soc. (2) 37, 383–401 (1934)], MacIntyre [ibid. 39, 282–294 (1935); Quart. J. Math., Oxford Ser. (1) 9, 182–184 (1938)] et Valiron [C. R. Acad. Sci. Paris 204, 33–35 (1937)]. Les auteurs donnent ici la définition générale précise suivante. Soient un ensemble

S de cercles $C(\alpha, R)$, $|z-\alpha|<R$ contenus dans un domaine d'holomorphie d'une fonction $f(z)$ et η un nombre réel de l'intervalle $(0, 1)$; $M(\alpha, R; \eta)$ et $m(\alpha, R; \eta)$ désignant le maximum et le minimum de $|f(z)|$ dans le cercle fermé $|z-\alpha|\leqq R\eta$ déduit de $C(\alpha, R)$, on appelle $L(\alpha, R; \eta)$ le plus petit des deux nombres

$$\log M(\alpha, R; \eta)/\log m(\alpha, R; \eta), \quad M(\alpha, R; \eta)/m(\alpha, R; \eta).$$

Si à S et à chaque η considéré correspond un nombre fini $A(\eta)$ tel que $L(\alpha, R; \eta)<A(\eta)$, l'ensemble des cercles S constitue un ensemble de plaines (flatness regions) de $f(z)$. S'appuyant sur des résultats antérieurs de Mandelbrojt [J. Math. Pures Appl. (9) **8**, 173–195 (1929)], les auteurs montrent que la condition nécessaire et suffisante pour qu'un ensemble S soit un ensemble de plaines est que l'ensemble des fonctions $f(\alpha+R\zeta)$, $|\zeta|<1$, forme une famille normale de Montel. Ils donnent ensuite des conditions suffisantes pour qu'un ensemble de plaines de $f(z)$ soit ensemble de plaines pour la dérivée $f'(z)$, puis pour la dérivée d'ordre $k>1$. Dans ce second cas $(k>1)$ les auteurs se bornent au cas où l'ensemble S est dénombrable. *G. Valiron.*

Bose, S. K. A note on the derivatives of integral functions. Ganita 1, 11–12 (1950).

À la suite d'une remarque faite dans l'analyse d'un mémoire antérieur [J. Indian Math. Soc. (N.S.) **10**, 77–80 (1946); ces Rev. **9**, 276], l'auteur complète ainsi son théorème: Si $M^{(q)}(r)$ désigne le maximum pour $|z|=r$ du module de la dérivée d'ordre q d'une fonction entière $f(z)$, la suite $M(r)$, $M^{(1)}(r)$, $\cdots$, $M^{(q)}(r)$, est croissante à partir d'une valeur $r_0(f)$ de r pourvu que l'ordre inférieur de $f(z)$ soit supérieur à 1. La démonstration repose sur une inégalité de Vijayaraghavan [J. London Math. Soc. **10**, 116–117 (1935)]. *G. Valiron* (Paris).

Erdös, P. On a theorem of Rådström. Proc. Amer. Math. Soc. **2**, 205–206 (1951).

The author proves that if $f(z)=\sum a_nz^n$ is entire, at least of order $p=1$ and infinite type, then there exist w_n, $|w_n|=1$, such that $k(z)=\sum w_na_nz^n$ has the origin as a limit of the zeros of its successive derivatives. This is a best possible improvement of a theorem of Rådström [Proc. Nat. Acad. Sci. U. S. A. **35**, 399–404 (1949); these Rev. **11**, 22] who required $p>1$. *R. C. Buck* (Madison, Wis.).

Agmon, Shmuel. Functions of exponential type in an angle and singularities of Taylor series. Trans. Amer. Math. Soc. **70**, 492–508 (1951).

Theorems of Cartwright [Quart. J. Math., Oxford Ser. (1) **7**, 46–55 (1936)] and Duffin and Schaeffer [Amer. J. Math. **67**, 141–154 (1945); these Rev. **6**, 148] on functions bounded at a sequence of points are obtained by novel methods including the use of normal families. A notable generalisation is achieved by defining a certain majorant $q(x)$ from the sequence $f(\lambda_n)$ so that the condition $|f(\lambda_n)|<K$ can be dropped while the conclusion $|f(x)|\leqq CK$ is replaced by $|f(x)|\leqq Cq(x)$. These results are applied to prove the author's gap theorem, generalisation of Szegö's theorem [C. R. Acad. Sci. Paris **226**, 1497–1499, 1673–1674, 1875–1876 (1948); these Rev. **9**, 576] and a converse of Fabry's theorem [cf. Macintyre and Wilson, J. London Math. Soc. **16**, 220–229 (1941); these Rev. **4**, 7, result (a)]. *A. J. Macintyre* (Aberdeen).

Yu, Chia-Yung. Sur les droites de Borel de certaines fonctions entières. Ann. Sci. École Norm. Sup. (3) **68**, 65–104 (1951).

The first part of this paper deals with entire functions defined by Dirichlet series. Representative results have been stated in two notes [C. R. Acad. Sci. Paris **228**, 641–643, 1833–1835 (1949); these Rev. **11**, 169]. The second part deals with functions defined by Laplace-Stieltjes transforms and with Dirichlet series in which the exponents are complex numbers. When the transform is "nearly" a Dirichlet series the results are similar to those of the first part. *R. P. Boas, Jr.* (Evanston, Ill.).

Tsuji, Masatsugu. On Borel's directions of meromorphic functions of finite order. I. Tôhoku Math. J. (2) **2**, 97–112 (1950).

L'auteur obtient les théorèmes connus sur les directions de Borel des fonctions méromorphes, et en complète certains, en utilisant non plus les résultats de Nevanlinna, mais ceux d'Ahlfors [Acta Math. **65**, 157–194 (1935)]. Il s'appuye sur les deux conséquences suivantes de la théorie d'Ahlfors: (1) Si $w(z)$ est méromorphe pour $|z|<1$ et si le nombre des zéros distincts de $[w(z)-a_1][w(z)-a_2][w(z)-a_3]$ dans $|z|<1$ est au plus égal à n, on a, pour $r<1$, $S(r)\leqq n+A/(1-r)$ où A est une constante dépendant seulement de a_1, a_2, a_3; $S(r)$ est la moyenne utilisée par Ahlfors. (2) Si $w(z)$ est méromorphe dans un angle Δ_0 ($|\arg z|\leqq\alpha_0$) et si Δ, $|\arg z|\leqq\alpha<\alpha_0$, est un angle intérieur à Δ_0, on a, pour $\lambda>1$,

$$T(r, \Delta)\leqq 3\sum_{i=1}^{3} N(\lambda r, a_i, \Delta_0)+A(\log r)^2,$$

où A ne dépend que de λ, α, α_0, α_1, α_2, α_3 et où $T(r, \Delta)$ et $N(r, a_i, \Delta_0)$ sont les fonctions relatives à l'angle Δ ou Δ_0 et à la valeur a_i. Parmi les compléments donnés par l'auteur, le théorème 3, p. 101, est contenu dans les résultats connus sur les cercles de remplissage [voir par exemple, le rapporteur, Directions de Borel des fonctions méromorphes, Memor. Sci. Math., no. 89, Gauthier-Villars, Paris, 1938]; le théorème 5 complète et précise un résultat du rapporteur [Bull. Sci. Math. (2) **59**, 298–320 (1935)]. Le théorème de la p. 104 relatif aux domaines couverts schlicht avait été signalé en partie par le rapporteur [C. R. Acad. Sci. Paris **194**, 1792 (1935)] et Ahlfors [Acta Soc. Sci. Fennicae. Nova Ser. A. **2**, no. 2, 1–17 (1933)]. Signalons enfin que Dufresnoy [Ann. Sci. École Norm. Sup. (3) **59**, 187–209 (1942); ces Rev. **6**, 149] a aussi appliqué systématiquement les résultats d'Ahlfors à l'étude des cercles de remplissage, mais sans donner de résultats explicites dans le cas de l'ordre nul. *G. Valiron* (Paris).

Tsuji, Masatsugu. On Borel's directions of meromorphic functions of finite order. II. Kōdai Math. Sem. Rep. **1950**, 96–100 (1950).

L'auteur complète les résultats du mémoire I [voir l'analyse ci-dessus]. Il précise la valeur du coefficient A figurant dans son théorème 2. Il démontre ensuite un théorème 3, analogue à 2 mais relatif à un secteur circulaire $|z|<1$, $|\arg z|\leqq\alpha_0$, ce qui lui sert à retrouver des théorèmes de Valiron sur les fonctions méromorphes dans le cercle $|z|<1$ [voir, le rapporteur, Bull. Sci. Math. (2) **56**, 10–32 (1932)] relatifs à l'existence de points de Borel ou de Picard. Il fait ensuite une étude analogue en considérant non plus le recouvrement de points, mais le recouvrement de domaines. Au théorème 1 du mémoire I correspond le théorème 7 donnant une borne de $S(r)$ en fonction du nombre de re-

couvrements simples de trois domaines disjoints. Il montre ainsi l'existence de directions (radiales) d'Ahlfors et introduit aussi à ce même point de vue des cercles analogues aux cercles de remplissage. De ces résultats, il faut encore rapprocher les indications données par Valiron, Ahlfors, et Dufresnoy dans les travaux cités dans l'analyse ci-dessus.

G. Valiron (Paris).

Tsuji, Masatsugu. On Borel's directions of meromorphic functions of finite order. III. Kōdai Math. Sem. Rep. **1950**, 104–108 (1950).

L'auteur continue à exposer les applications des résultats obtenus par la méthode d'Ahlfors dans ses mémoires I et II [voir les deux analyses ci-dessus]. Il établit ce théorème: Soient $f(z)=[w(z)g_1(z)+g_2(z)]/[w(z)g_3(z)+g_4(z)]$ où $w(z)$ et les $g_i(z)$ sont méromorphes à distance finie,

$$T(r,g)=\sum_{i=1}^{4}T(r,g_i),$$

Δ_0 l'angle $|\arg z|\leqq\alpha_0$, Δ l'angle $|\arg z|\leqq\alpha<\alpha_0$ et $S(r,f;\Delta_0)$ la moyenne d'Ahlfors relative à l'angle Δ_0, on a

$$S(r,f;\Delta)\leqq 27S(2r,w;\Delta_0)+O\left(\int_1^{2r}T(r,g)r^{-1}dr\right).$$

De ce théorème, il déduit des résultats de Biernacki [Acta Math. **56**, 197–204 (1930)] et Rauch [J. Math. Pures Appl. (9) **12**, 109–171 (1933)] sur les directions de Borel. Si $f(z)$ est méromorphe d'ordre fini $\rho>0$, il existe une direction J telle que, Δ étant un angle arbitraire contenant J, la série $\sum|z_\nu|^{-\tau}$ diverge pour $\tau<\rho$, les z_ν étant les zéros distincts de $f(z)=g(z)$ appartenant à Δ et $g(z)$ une fonction méromorphe quelconque d'ordre inférieur à ρ, différente de deux fonctions exceptionnelles possibles. Si $f(z)$ est du type divergent de l'ordre ρ, la série (1) diverge pour $\tau=\rho$ quelle que soit $g(z)$ d'ordre inférieur à ρ, ou d'ordre ρ et du type convergent, distincte de deux fonctions exceptionnelles possibles.

G. Valiron (Paris).

Tsuji, Masatsugu. On a regular function which is of constant absolute value on the boundary of an infinite domain. Tôhoku Math. J. (2) **3**, 24–38 (1951).

The author deals with the distribution of values of a function $w(z)$, analytic on an infinite domain Δ whose boundary Γ consists of a countable number of analytic curves, and subject to $|w(z)|<R$ in Δ, $|w(z)|=R$ on Γ. The conventional definition of the quantities $m(r,a)$ and $N(r,a)$ is modified as as follows. The metric $(w,0)=2R|w|/(R^2+|w|^2)$ is introduced in $|w|\leqq R$. Denoting $U_a(w)=R^2(w-a)/(R^2-\bar{a}w)$, the distance between a and b is given by $(a,b)=(U_a(b),0)$. Then $m(r,a)$ is defined as $(1/2\pi)\int\log\{1/(w(re^{i\theta}),a)\}d\theta$ where the integral is taken along the intersection of $|z|=r$ and Δ. In the definition of $N(r,a)$ the a-points are counted in the intersection of $|z|\leqq t$ $(\leqq r)$ and Δ. The author shows that then $T(r,a)=m(r,a)+N(r,a)$ is an increasing convex function of $\log r$. Relations among $T(r,a)$, $\sum N(r,a_i)$ and the corresponding characteristic $T(r)$ are derived, using the above metric. Certain extensions are indicated.

L. Sario (Princeton, N. J.).

Rahmanov, B. N. On the theory of univalent functions. Doklady Akad. Nauk SSSR (N.S.) **78**, 209–211 (1951). (Russian)

Let K be the class of functions $f(z)=z+\cdots$ which are regular, univalent, and convex in $|z|<1$. Starting from a lemma which the author states is obvious it is proved that, if $f(z)\varepsilon K$, $-\frac{1}{2}\pi\leqq\alpha\leqq\frac{1}{2}\pi$, then

$$\Phi_\alpha(z)=(f(z)+e^{i\alpha}zf'(z))/(1+e^{i\alpha})=z+\beta_2z^2+\beta_3z^3+\cdots$$

is univalent in $|z|<1$. The author notes that if $\alpha=\frac{1}{2}\pi$ and $f(z)=z/(1-z)$, then $|2\beta_3-\beta_2^2|>2$, so that $z/(1-z)^2$ does not furnish the maximum for an expression of this sort in the family of univalent functions. Bounds are stated for the radius of convexity and the radius of starlikeness of $\Phi_0(z)$, if $f(z)\varepsilon K$. Let $F(z)=z+\cdots$ be regular and univalent, and suppose $|\arg(zF'(z)/F(z))|\leqq\frac{1}{4}\pi$ in $|z|<1$. A number of inequalities involving $F(z)$ are given without proof. In each case equality occurs for $F(z)=2ze^{\arcsin z}/[1+(1-z^2)^{\frac{1}{2}}]$. It is proved that if $f(z)\varepsilon K$, and $0<\alpha\leqq\frac{1}{2}\pi$, then $f(e^{i\alpha}z)-f(e^{-i\alpha}z)$ is starlike in $|z|<1$. *A. W. Goodman* (Lexington, Ky.).

Ilieff, Ljubomir. Zur Theorie der schlichten Funktionen. Annuaire [Godišnik] Univ. Sofia. Fac. Sci. Livre 1. **45**, 115–135 (1949). (Bulgarian. German summary)

The author considers the class S of functions

$$f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$$

which are regular and schlicht in $|z|<1$, and also the classes S_k $(k=1,2,\cdots)$ of functions

$$f_k(z)=\{f(z^k)\}^{1/k}=z+a_{k+1}z^{k+1}+a_{2k+1}z^{2k+1}+\cdots,$$

$f\varepsilon S$. From an inequality of G. M. Goluzin [Mat. Sbornik N.S. **19**(61), 183–202 (1946); these Rev. **8**, 325], he obtains upper and lower bounds for $|f_k'(z)|$, $|(f_k(z_1)-f_k(z_2))/(z_1-z_2)|$, $|zf_k'(z)/f_k(z)|$ and, following a method due to Szegö, finds upper bounds for the radii of circles $|z|<\rho$ inside which all partial sums of the power series are schlicht. Some of the author's results are also given by Jenkins [see the following review]. *D. C. Spencer* (Princeton, N. J.).

Jenkins, James A. On an inequality of Golusin. Amer. J. Math. **73**, 181–185 (1951).

Goluzin [Mat. Sbornik N.S. **18**(60), 167–179 (1946); **19**(61), 183–202 (1946); these Rev. **7**, 515; **8**, 325] has proved that, if $F(\zeta)=\zeta+a_0+a_1/\zeta+\cdots$ is meromorphic and schlicht for $|\zeta|>1$, then $|F(\zeta_1)-F(\zeta_2)|/|\zeta_1-\zeta_2|\geqq 1-r^{-2}$, where $|\zeta_1|=|\zeta_2|=r$, $r>1$. The author begins by pointing out that Goluzin's inequality is an immediate consequence of an earlier theorem of H. Grötzsch [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. **81**, 38–47 (1929)], namely: Let D be a domain of the ζ-plane containing $\zeta=\infty$ and of finite connectivity, and let ζ_1, ζ_2 be finite interior points of D. Then $|F(\zeta_1)-F(\zeta_2)|$ attains its minimum within the class of functions F of the above type for the function which maps D onto the plane slit along arcs of the confocal family of hyperbolas having the images of ζ_1 and ζ_2 as foci. Let $f(z)=z+c_2z^2+\cdots$ be regular and schlicht in $|z|<1$, and let $s_n(z)=z+c_2z^2+\cdots+c_nz^n$ be the nth partial sum. The author applies Goluzin's inequality to obtain a simple proof of Szegö's result that $s_n(z)$ is schlicht for $|z|<\frac{1}{4}$ $(n=1,2,\cdots)$. Next the author, improving a result of V. Levin, proves that for given n the partial sum $s_n(z)$ is schlicht for $|z|\leqq 1-\beta_n/n$, $\beta_n=4\log n-\log(4\log n)$. Finally he shows that if

$$f(z)=z+a_3z^3+a_5z^5+\cdots$$

is an odd schlicht function, then the partial sums are schlicht for $|z|<3^{-\frac{1}{2}}$ (a conjecture of K. Joh). [Reviewer's remark: Ilieff [see the preceding review], also using Goluzin's inequality, shows that $s_n(z)$ is schlicht for $|z|<1-\beta_n/n$, $\beta_n=4\log n$, provided that $n\geqq 15$, and furthermore proves that the partial sums of an odd schlicht function are schlicht for $|z|<3^{-\frac{1}{2}}$.] *D. C. Spencer* (Princeton, N. J.).

Warschawski, S. E. On conformal mapping of regions bounded by smooth curves. Proc. Amer. Math. Soc. **2**, 254–261 (1951).

Soit C une courbe simple fermée rectifiable, pourvue d'une tangente continue dont l'angle $\alpha(s)$, considéré comme fonction de l'arc s, satisfait à $|\alpha(s\pm t)-\alpha(s)|\leqq\beta(t)$. On suppose que le rapport du plus petit arc P_1P_2 de C à la corde P_1P_2 qui le sous-tend est borné par k, que le diamètre de C est borné par D, et que la région R intérieure à C contient le cercle de rayon $\sigma>0$ centré à l'origine. Alors si $w=f(z)$ représente conformément le cercle $|z|<1$ sur R avec $f(0)=0$, pour tout $p>0$, il existe une constante A_p dépendant seulement de p, k, D, σ, et $\beta(t)$, telle que

$$\left\{\frac{1}{2\pi}\int_0^{2\pi}|f'(\rho e^{i\theta})|^{\pm p}d\theta\right\}^{1/p}\leqq A_p\qquad[0\leqq\rho<1].$$

Cette inégalité précise un résultat qualitatif précédemment acquis par l'auteur, et permet, quel que soit $\delta>0$, de déterminer une constante B, dépendant seulement de δ, k, D, σ, et $\beta(t)$, telle que pour $|z_0|=1$, $|z|<1$, on ait:

$$B^{-1}|z-z_0|^{1/(1-\delta)}\leqq|f(z)-f(z_0)|\leqq B|z-z_0|^{1-\delta}.$$

J. Lelong-Ferrand (Lille).

Lavrent'ev, M. The general problem of the theory of quasi conformal mappings of plane regions. Amer. Math. Soc. Translation no. **46**, 53 pp. (1951).

Translated from Mat. Sbornik N.S. **21(63)**, 285–320 (1940); these Rev. **10**, 290.

Springer, George. Pseudo-conformal transformations onto circular domains. Duke Math. J. **18**, 411–424 (1951).

Étude de la représentation analytique complexe biunivoque (pseudo-conforme) d'un domaine D de $\mathfrak{C}^2$ sur un domaine C de Reinhardt: $|z_1|<\rho_1=G(|z_2|)$ de centre $O[z_1=0, z_2=0]$. En utilisant l'invariant $J_D(z_1, z_2)$ déduit du noyau de Bergmann, et la structure topologique des surfaces $J_C=\text{cte.}$ au voisinage de O, on obtient des conditions nécessaires et suffisantes pour que la représentation soit possible; l'auteur fait là une étude plus précise d'une possibilité signalée par S. Bergmann [Mémor. Sci. Math., fasc. 108, Gauthier-Villars, Paris, 1948, cf. pp. 47, 48; ces Rev. **11**, 344]. Il montre de plus que, de la détermination du point $P_0\varepsilon D$ homologue du centre O de C, on déduit explicitement les équations de la représentation. Soit

$$J_C=B_0+B_{10}|z_1|^2+B_{01}|z_2|^2+\cdots;$$

étude des deux cas $B_{10}B_{01}<0$ et $B_{10}B_{01}>0$. Étude plus précise pour les domaines $C(A)$: $|z_2|^2+|z_1|^2+A|z_1|^4<1$, $A>0$, pour lesquels la seconde inégalité est réalisée (J_C a un maximum relatif isolé unique en O). L'auteur donne également un théorème de distorsion relatif à la représentation sur E_λ: $|z_2|^2<\exp(-\lambda|z_1|^2)$ d'un domaine D pour lequel on suppose $E_\mu\subset D\subset E_\nu$. *P. Lelong* (Lille).

Reade, Maxwell O. A theorem of Féderoff. Duke Math. J. **18**, 105–109 (1951).

Un théorème de Féderoff exprime que $f(z)$ finie continue est holomorphe dans le domaine circulaire unité $\mathfrak{D}$ par la nullité de l'intégrale en aire de $(z-z_0)f(z)$ sur tout cercle fermé D contenu dans $\mathfrak{D}$. Pour une fonction finie continue la condition d'être monogène aréolaire c'est-à-dire possédant des dérivées premières continues telles que le $\partial/\partial x+i\partial/\partial y$ soit holomorphe équivaut à la nullité des intégrales de $(z-z_0)f(z)$ ou de $(z-z_0)^2f(z)$ respectivement sur la circonférence ou l'aire de tout D. Si l'on remplace D par un polygone régulier de n cotés, ces conditions pour une fonction continue caractérisent respectivement les polynomes de degré $\leqq n-2$ ou certaines fonctions monogènes aréolaires.

M. Brelot (Grenoble).

Hvedelidze, B. V. On the problem of linear conjunction in the theory of analytic functions. Doklady Akad. Nauk SSSR (N.S.) **76**, 177–180 (1951). (Russian)

Let C be a finite collection of disjoint, simple, open arcs, the tangent making with a fixed direction an angle satisfying a Hölder condition. The plane, cut along C, is denoted by P. If $\varphi(t)$ is given on C and $|\varphi(t(s))|^p$ ($t=t(s)$ is the equation of C) is summable, $\varphi(t)$ belongs to $L^p(C)$. The class $A_{n,p}(C)$ consists of functions $f(z)$ such that: (1) $f(z)$ is analytic in P; (2) $f(z)=O(z^n)$ at infinity; (3) the nontangential limits $f^+(t)$, $f^-(t)$ exist a.e. (almost everywhere) on C and are in $L^p(C)$; (4) $S(f^+-f^-)=f^++f^-+\gamma_n$ (γ_n an arbitrary polynomial of order n), with $\pi iS\varphi=\int_C\varphi(t)(t-t_0)^{-1}dt$, t_0 on C (sense of principal values). The author solves the problem of finding $\Phi(z)$ in $A_{-1,p}(C)$, such that

$$\Phi^+(t)=G(t)\Phi^-(t)+g(t)$$

a.e. on C, $G(t)$, $g(t)$ given on C. This problem, previously considered by N. I. Mushelišvili in the case of $g(t)$, $G(t)$ of Hölder class and the solution Φ piecewise analytic (in the sense of Mushelišvili), is now solved under considerably more general conditions on g, G, Φ. [Cf. the same Doklady N.S. **76**, 367–370 (1951); these Rev. **12**, 832.]

W. J. Trjitzinsky (Urbana, Ill.).

Baganas, Nicolas. Un critère de normalité d'une famille de fonctions algébroïdes. C. R. Acad. Sci. Paris **232**, 1534–1536 (1951).

Let $\{u_j(z)\}$ be a sequence of algebroid functions of n branches defined in a domain D by means of the equations

$$\text{(A)}\quad F_j(u, z)=f_0{}^j(z)u^n+f_1{}^j(z)u^{n-1}+\cdots+f_n{}^j(z)=0,\qquad(j=1, 2, \cdots),$$

where $f_k{}^j(z)$, $k=0, 1, \cdots, n$, are regular in D and do not vanish for the same value of z. To each equation $F_j(u, z)=0$, and for each z in D let $P_j=P_j[f_0{}^j(z), \cdots, f_n{}^j(z)]$ be the point in complex projective n-space, and let the metric in this space be

$$|P_iP_j|=\min_{0\leqq\theta\leqq2\pi}\sum_{k=0}^{n}\left|\frac{\alpha_k{}^i}{N_i}-\frac{\alpha_k{}^j}{N_j}e^{i\theta}\right|,$$

where $P_i=P_i[\alpha_0{}^i, \alpha_1{}^i, \cdots, \alpha_n{}^i]$ and $N_i=(\sum_{k=0}^n|\alpha_k{}^i|^2)^{\frac{1}{2}}\neq0$. The sequence (A) is said to "converge uniformly" in D if, for $\epsilon>0$, there exists an integer $N(\epsilon)$ independent of z, such that for $i, j>N(\epsilon)$, $|P_iP_j|<\epsilon$. It is asserted that, if the sequence (A) "converges uniformly" in D, then the corresponding sequence $\{u_j(z)\}$ converges uniformly (in the ordinary sense) to an algebroid function of n branches, and conversely. The concept of normality being the usual one, the main assertion is that the algebroid functions of n branches in D which do not assume in D any of the values of a given algebraic function having at least $2n+1$ distinct branches form a normal family in D. *A. J. Lohwater.*

Ou, Vincent Tchen-yang. Valeurs déficientes d'une fonction algébroïde. C. R. Acad. Sci. Paris **232**, 2073–2075 (1951).

Consider an algebroid function $F(x)$, defined by

$$\sum_{i=0}^{\nu}A_i(x)F^{\nu-i}=0,$$

with meromorphic coefficients $A_i(x)$ ($A_0(x)\equiv 1$). The author announces the following results: (1) There are at most a countable number of defective values of $F(x)$; (2) the sum of the defects is at most 2ν. Some inequalities concerning the relation of the characteristic function to the branch-points over $|x|<r$ and to the zeros of $F(x)-\alpha_i$ $(i=1, \cdots, q)$ are given.

L. Sario (Princeton, N. J.).

Tôyama, Hiraku. Zur Theorie der hyperabelschen Funktionen. IV. Proc. Japan Acad. **22**, 188–194 (1946).

Tôyama, Hiraku. Verallgemeinerung des Abelschen Integrals und Periodenrelationen. Proc. Japan Acad. **22**, 238–241 (1946).

Tôyama, Hiraku. Über eine nicht-Abelsche Theorie der algebraischen Funktionen. Bull. Tokyo Inst. Tech. Ser. B [illegible], 253–321 (1950).

Let $\bar{F}$ be the universal covering surface of the Riemann surface F belonging to a function field K where nonunit signatures are given at a finite number of points p. Suppose that G is the group of covering transformations of $\bar{F}/F$. In this memoir the author presents a summary of his studies on matrices of meromorphic functions of $\bar{F}$ and the representation theory of G. The work is largely based upon A. Weil's fundamental paper [J. Math. Pures Appl. (9) **17**, 47–87 (1938)]. The following brief summary of the various chapters will have to suffice as a short indication of the work since highly technical details are necessary for the description of the individual results. Chapter 1. Generalized divisors (a divisor of degree r is a collection $U_p\theta_p$ of left cosets of the groups of regular matrices of degree r modulo the group of p-adic units U_p, which are invariant for the cyclic groups assigned to the branch points). Chapter 2. The theorem of Riemann-Roch-Weil (with a formal simplification of Weil's original proof using the calculus of residues). Chapter 3. Representation theory of the fundamental group (the algebraic varieties determined by the matricial representations of G). Chapter 4. The hyper-Abelian integral (multiplicative integrals of Poincaré and Volterra on Riemann surfaces, period relations). Chapter 5. Classes of divisors (normalization of divisors in almost all classes, logarithmic differentials). Chapter 6. The non-Abelian principal divisor theorem (detailed study of logarithmic differentials). Chapter 7. Unitary representations (study of Weil's "kernels" of representations, counting of parameters, existence of classes of representations which do not contain unitary representations). Chapter 8. The duality theorem. The author emphasizes particularly the structure of the various representations of the fundamental group G. This work requires a careful discussion of the normal forms of matrices on $\bar{F}$ and the additive expansion of the logarithmic differential dI of a simple divisor θ ($\theta dI\theta^{-1}-d\theta\theta^{-1}$ is everywhere finite on $\bar{F}$ [see Weil, loc. cit., p. 77; and chapters 5, 6 above]). In order to describe the relations between divisor classes and representations of the fundamental group preference over Poincaré's theory of zeta-Fuchsian functions is given to the theory of linear differential equations of Riemann surfaces as a tool in existence proofs [see also Weil, loc. cit., p. 74]. The duality theorem is phrased as a statement characterizing certain mappings of a field which is obtained from K by adjunction of the elements belonging to zeta-Fuchsian functions of a unitary representation (everywhere dense in the appropriate unitary group), by means of the analyticity as elements of G.

O. F. G. Schilling (Chicago, Ill.).

Tôyama, Hiraku. Über den nicht-Abelschen Hauptdivisorsatz. Proc. Japan Acad. **24**, no. 2, 8 pp. (1948).

The content of this paper corresponds to parts of Chapter VI of the last paper reviewed above. *O. F. G. Schilling.*

Takahashi, Shin-ichi. Univalent mappings in several complex variables. Ann. of Math. (2) **53**, 464–471 (1951).

Le résultat essentiel de cet article consiste en une transposition au cas des transformations analytiques de l'espace de n variables complexes, du théorème démontré par S. Bochner pour les transformations harmoniques dans l'espace de n variables réelles [Bull. Amer. Math. Soc. **52**, 715–719 (1946); ces Rev. **8**, 204]. Ce théorème lui-même généralisait le fameux théorème d'A. Bloch: Une fonction $f(z)$ d'une variable complexe z, analytique pour $|z|<1$, telle que $|f'(0)=1$, applique biunivoquement un certain ensemble ouvert de $|z|<1$ sur un disque de rayon plus grand qu'une constante universelle. Dans la généralisation de Bochner, la transformation harmonique est supposée "presque conforme", i.e., il existe une constante K telle que le rapport du plus grand axe au plus petit axe de l'"ellipsoïde de dilatation" soit, en chaque point, $\leq K$. L'auteur établit un résultat analogue pour les transformations analytiques de n variables complexes, grâce à une hypothèse analogue à celle de Bochner, quoique légèrement affaiblie. La démonstration suit celle de Bochner; une majoration de la constante de Bloch est donnée en fonction de n et de K. Le début de l'article est consacré à des énoncés préparatoires, de caractère plus topologique qu'analytique. *H. Cartan* (Paris).

Rothstein, Wolfgang. Über die Fortsetzung analytischer Flächen. Math. Ann. **122**, 424–434 (1951).

Ziel der Arbeit ist die Verallgemeinerung eines kürzlich vom gleichen Verf. aufgestellten Satzes über die Fortsetzbarkeit $(2n-2)$-dimensionaler analytischer Flächen im Raume R_{2n} von n komplexen Veränderlichen ($n\geqq 3$) [siehe Math. Ann. **121**, 340–355 (1950); diese Rev. **11**, 652]. Im wesentlichen wird folgendes bewiesen: $\mathfrak{B}$ sei ein schlichter, beschränkter, von endlich vielen Hyperebenenstücken begrenzter Bereich des R_6; $\bar{\bar{\mathfrak{F}}}$ ein 4-dimensionales analytisches Flächenstück und $\bar{S}$ eine zusammenhängende Schnittmenge von $\bar{\bar{\mathfrak{F}}}$ mit dem Rande $\mathfrak{R}$ von $\mathfrak{B}$, die ohne Aufgabe des Zusammenhangs nicht erweitert werden kann. $\bar{\bar{\mathfrak{F}}}_S$ sei der in einer Umgebung $\mathfrak{U}_\epsilon(\mathfrak{R})$ gelegene Teil von $\bar{\bar{\mathfrak{F}}}$, der $\mathfrak{R}$ genau in $\bar{S}$ schneidet; ferner sei $\bar{S}$ überall 3-dimensional und zerlege $\bar{\bar{\mathfrak{F}}}_S$ in zwei nur über $\bar{S}$ zusammenhängende Teile. Dann lässt sich $\bar{\bar{\mathfrak{F}}}_S$ in ein singularitätenfreies analytisches Flächenstück fortsetzen, das nur von $\bar{S}$ berandet wird. In der oben zitierten Arbeit wurde der Satz für schlichte beschränkte Regularitätsbereiche bewiesen; das von $\bar{S}$ berandete Flächenstück verläuft dann notwendig ganz in $\mathfrak{B}$. Im allgemeineren, hier behandelten Falle ist das nicht immer der Fall, und das Flächenstück $\bar{\bar{\mathfrak{F}}}$ kann den Rand $\mathfrak{R}$ noch in von $\bar{S}$ verschiedenen Mannigfaltigkeiten schneiden. Die analytischen Flächen werden wie in der ersten Arbeit durch algebraische Verteilungen (Cousinsche Verteilungen zweiter Art) gegeben. Es sei hinzugefügt, dass die bewiesene Aussage auch auf beliebiges $n>3$ übertragen werden kann und ferner, dass sie eng mit einem Satz von H. Behnke zusammenhängt, nach welchem eine endlichblättrige Funktion $f(z_1, \cdots, z_n)$, die auf einer geschlossenen $(2n-1)$-dimensionalen Fläche S regulär und eindeutig ist (f sei zudem von einer Seite von S zur andern nur über die auf S festgesetzten Funktionselemente fortsetzbar), nach einer Seite von S unbeschränkt fortgesetzt werden kann. *P. Thullen* (Panamá).

Lockot, Georg, und Schmidt, Hermann. Über Nullgebilde analytischer Funktionen zweier Veränderlichen, die in singulären Stellen münden. I. Durch ganze Potenzen asymptotisch approximierbare Nullgebilde. Math. Ann. 122, 411–423 (1951).

Gegeben sei eine analytische, durch eine Hartogssche Reihe $\sum x^\nu f_\nu(y)$ dargestellte Funktion $f(x, y)$, die in jeder Umgebung von $(0, y_0)$ reguläre Punkte aufweise, in $(0, y_0)$ selbst aber singulär werden kann. Verf. betrachten Nullgebilde $y=y(x)$ der gegebenen Funktion, für welche noch $y\to y_0$, falls $x\to x_0$, und stellen sich die Aufgabe, Bedingungen für die Existenz solcher Nullgebilde aufzuweisen und Näherungspolynome aufzustellen, welche diese Nullgebilde in einer Teilumgebung von $(0, y_0)$ asymptotisch darstellen. Es sei erinnert, dass in dem für $f(x, y)$ möglicherweise singulären Punkt $(0, y_0)$ der Weierstrass'sche Vorbereitungssatz nicht angewandt werden kann. Nachdem einer der Verfasser in einer früheren Arbeit [H. Schmidt, Math. Z. 43, 533–552 (1938)] die gestellte Aufgabe gelöst hatte unter der Voraussetzung, dass $f(x, y)$ in einem Zylindergebiet $\mathfrak{G}=\{|x|<\xi, y\epsilon\mathfrak{T}\}$ konvergiere, wobei ξ fest und $\mathfrak{T}$ ein Winkelraum mit dem Scheitel $y_0=1$ sei, werden in vorliegender Arbeit die gewonnenen Ergebnisse auf allgemeinere Gebiete übertragen, indem zugelassen wird, dass der Konvergenzradius der gegebenen Hartogsschen Reihe mit $y\to y_0$ gegen Null geht und $\mathfrak{T}$ nicht nur ein Winkelraum beliebiger Lage, sondern auch das Zwischengebiet zweier sich berührender Kurven sein darf. Allerdings werden zusätzliche Annahmen in dem Sinne gemacht, dass der Konvergenzradius ξ nicht zu schnell gegen Null gehe und der Grad der Berührung der beiden $\mathfrak{T}$ bestimmenden Kurven nicht zu hoch sei. *P. Thullen* (Panamá).

Nisigaki, Hisami. Geometrical properties of quaternion-functions. Mem. Fac. Sci. Eng. Waseda Univ. 14, 5–6 (1950).

The paper is an announcement of results without proof. The differential equation in real quaternions

$$dY=A(dX)B+C(dX)D$$

is solved, the solution containing four arbitrary analytic functions of two complex variables. Next consider a hypersurface in Euclidean 8-space defined by

$$F(z_1, \cdots, z_4, \bar{z}_1, \cdots, \bar{z}_4)=0,$$

the z_i and $\bar{z}_i$ being complex variables and their conjugates. The author considers the class Γ of all $F=0$ satisfying certain differential equations and boundary conditions (which are too lengthy to enumerate here), and determines all origin-preserving transformations which carry each member of Γ into another. The same problem is solved in the case where the dimensions are reduced by the additional restrictions $z_3=\bar{z}_1$, $z_4=\bar{z}_2$. *I. Niven* (Eugene, Ore.).

Theory of Series

Rogers, C. A. The transformation of sequences by matrices. Proc. London Math. Soc. (2) 52, 321–364 (1951).

Twenty one theorems give characterizations of complex-valued functions $a_1(\omega)$, $a_2(\omega)$, $\cdots$, defined for $\omega>0$, such that the transformation $\sum_{k=1}^\infty a_k(\omega)x_k$ has specified properties for each complex sequence x_k in a given class S. A class S of sequences is called a sequence space if it is linear, that is, if the sequence $z_k=\lambda x_k+\mu y_k$ belongs to S whenever λ and μ are complex constants and the sequences x_k and y_k belong to S. A sequence space is called normal if it contains z_k whenever it contains a sequence x_k for which $|z_k|\leqq|x_k|$, $k=1, 2, \cdots$. It is shown that if S is a normal sequence space and if, as $\omega\to\infty$, $\limsup_{n\to\infty}|\sum_{k=1}^n a_k(\omega)|x_k||=O(1)$ [or $o(1)$ or $o(\infty)$] for each sequence x_k in S, then $\sum_{k=1}^\infty|a_k(\omega)x_k|=O(1)$ [or $o(1)$ or $o(\infty)$] for each x_k in S; moreover the converse holds. If S is a normal sequence space and the functions $a_k(\omega)$ are real and $a_k(\omega)\to 0$ as $\omega\to\infty$ and $\sum_{k=1}^\infty|a_k(\omega)x_k|<\infty$ when x_k belongs to S and $\omega>\omega_0$, then $\liminf_{\omega\to\infty}\sum_{k=1}^\infty a_k(\omega)|x_k|\geqq 0$ [or $>-\infty$] for each sequence x_k of S if and only if $\limsup_{\omega\to\infty}\sum_{k=1}^\infty\{|a_k(\omega)|-a_k(\omega)\}|x_k|=0$ [or $<+\infty$] for each sequence x_k of S. Two theorems deal with total regularity, a complex regular transformation being called totally regular if $\lim_{\omega\to\infty}\liminf_{n\to\infty}\Re\{\sum_{k=1}^n a_k(\omega)x_k\}=+\infty$ and $\lim_{\omega\to\infty}\limsup_{n\to\infty}|\Im\{\sum_{k=1}^n a_k(\omega)x_k\}|=0$. Further theorems treat the behavior of sets of limit points and of Knopp's cores under regular transformations. It seems that $\Im|\{a_k(\omega)\}x_k|$ is misprinted for $|\Im\{a_k(w)\}x_k|$ in line 2 of p. 335 where a conjecture on cores is presented. As the author says, many of his results have been proved by various authors in the special cases when S is the space of all bounded sequences and the space of all sequences. *R. P. Agnew.*

Ogieveckiĭ, I. I. On S. N. Bernšteĭn's summation method. Doklady Akad. Nauk SSSR (N.S.) 76, 635–638 (1951). (Russian)

A series $\sum a_k$ is evaluable to s by the Bernstein method B_m of integer order m if $t_n\to s$ where $t_n=\sum_{k=0}^n a_k\cos^m[k\pi/(2n+1)]$. If $r>-1$, each series evaluable by the Cesàro method C_r is evaluable B_m to the same value when m is the least integer greater than or equal to r. A sequence s_n is evaluable MC to s if $t_n\to s$ where $t_n=(\sigma_n+\sigma_{n+1})/2$ and $\sigma_n=(s_0+s_1+\cdots+s_n)/n$. Karamata [Rec. Math. [Mat. Sbornik] N.S. 21(63), 13–24 (1947); these Rev. 9, 140] proved that B_1 and MC are equivalent and later [Math. Z. 52, 305–306 (1949); these Rev. 11, 347] gave a shorter direct proof not cited by the author. If $m=2, 3, 4, \cdots$ and $r\geqq 2$, then B_m and C_r include C_2 which is equivalent to the Hölder method H_2 which is stronger than MC which is equivalent to B_1; therefore both B_m and C_r are stronger than B_1. *R. P. Agnew.*

Hill, J. D. Note on a theorem in summability. Proc. Amer. Math. Soc. 2, 372–373 (1951).

The theory of linear bounded operators is used to prove the following theorem of Henstock [J. London Math. Soc. 25, 27–33 (1950); these Rev. 11, 429]. If s_n is a bounded sequence, then there is a countable set S of sequences of zeros and ones such that s_n is evaluable by each regular matrix method which evaluates each sequence of S.
R. P. Agnew (Ithaca, N. Y.).

Kuttner, B. A new method of summability. Proc. London Math. Soc. (2) 53, 230–242 (1951).

Let λ_n be a sequence for which $0<\lambda_0<\lambda_1<\cdots$ and $\lambda_n\to\infty$. A series $\sum a_n$ is evaluable by the new method (X, λ_n) to L if $X(x)\to L$ as $x\to\infty$ where

$$X(x)=\sum_{\lambda_n\leqq x}(e^{\lambda_n})\left(1-\frac{\lambda_n}{x}\right)^x a_n.$$

Letting $A(u)=\sum_{\lambda_n\leqq u}a_n$ gives $X(x)=\int_0^x(e^u)(1-u/x)^x dA(u)$. The main interest in the method (X, λ_n) lies in the fact that it is a straightforward Silverman-Toeplitz method, not involving iterated limits, which includes the Riesz methods (R, λ_n, k), $k=1, 2, 3, \cdots$, by which $\sum a_n$ is evaluable to L if

$A^{(k)}(u) \to L$ as $u \to \infty$ where $A^{(k)}(u) = ku^{-k}\int_0^u A(t)(u-t)^{k-1}dt$. Regions into which (X, λ_n) gives analytic extensions of Dirichlet series $\sum a_n \exp(-\lambda_n s)$ of type λ_n are given, for the case in which the series converges in a half-plane to a function $f(s)$, in terms of the singularities of $f(s)$. The fact that (X, λ_n) includes the Riesz methods (R, λ_n, k) suggests that (X, λ_n) may also include the Dirichlet series (or generalized Abel) method (A, λ_n) by which $\sum a_n$ is evaluable to L if $\sum_{n=0}^{\infty} a_n \exp(-\lambda_n s) \to L$ as $s \to 0+$. For the special case $\lambda_n = n$, it is shown that (X, λ_n) does not include (A, λ_n).

R. P. Agnew (Ithaca, N. Y.).

Tsuchikura, Tamotsu. Arithmetic means of subsequences. Tôhoku Math. J. (2) 2, 188–191 (1950).

H. Pollard and the reviewer [Bull. Amer. Math. Soc. 49, 924–931 (1943); these Rev. 5, 117] proved that if $\{S_n\}$ is summable $(C, 1)$ and if $\sum (S_k/k)^2 < \infty$ then almost all the subsequences of $\{S_n\}$ are also summable $(C, 1)$. Using the law of the iterated logarithm, the author improves this condition to $\sum_1^n S_k^2 = o(n^2/\log\log n)$ and shows that this is best possible with an example which obeys $\sum_1^n S_k^2 = O(n^2/\log\log n)$. Turning from measure to category, if $\{S_n\}$ is summable $(C, 1)$ but not convergent, then the summable subsequences form a set of first category.

R. C. Buck.

Timan, M. F. On (C, α, β)-summability of double series. Doklady Akad. Nauk SSSR (N.S.) 76, 647–649 (1951). (Russian)

Let $\alpha, \beta > -1$. If $\sum u_{mn}$ is evaluable (C, α, β) to S and the rows and columns of the (C, α, β)-transform satisfy stated conditions, then $\sum u_{mn}$ is restrictedly evaluable to S by the Abel power series method. Related results, some of which are known, are given.

R. P. Agnew (Ithaca, N. Y.).

Čelidze, V. G. On the transformation of double sequences. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 17, 61–94 (1949). (Russian. Georgian summary)

Several lemmas and theorems give conditions on a matrix a_{mnjk}, necessary or sufficient or both, to ensure that for each convergent sequence s_{mn} in a specified class the series in

$$\sigma_{mn} = \sum_{j,k=0}^{\infty} a_{mnjk} s_{jk}$$

converges for each $m, n = 0, 1, 2, \cdots$ and determines a double sequence σ_{mn} which converges (or converges restrictedly in the sense of C. N. Moore) to lim s_{mn}. One class of sequences s_{mn}, depending on functions $\varphi(x)$ and $\psi(x)$ for which $\varphi(x) \to \infty$ and $\psi(x) \to \infty$ as $x \to \infty$, consists of those convergent double sequences s_{mn} for which $\limsup_{m\to\infty} |s_{mn}|/\varphi(m) = \alpha_n < \infty$ for each n and $\limsup_{n\to\infty} |s_{mn}|/\psi(n) = \beta_m < \infty$ for each m. Another class is the subclass for which $\alpha_n = \beta_m = 0$ for each m and n. Each theorem involves 7 or 8 conditions on the matrix a_{mnjk}, some of which involve the functions $\varphi(x)$ and $\psi(x)$. The conditions resemble the conditions of G. M. Robison [Trans. Amer. Math. Soc. 28, 50–73 (1926)], where bounded convergent sequences are treated. A reference to C. N. Moore's treatment [Summable Series and Convergence Factors, Amer. Math. Soc. Colloq. Publ., vol. 22, New York, 1938] of this subject should be given.

R. P. Agnew (Ithaca, N. Y.).

Karamata, J. A theorem of Tauberian nature connected with known theorems of Hadwiger. Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka 198, 147–161 (1950). (Serbo-Croatian)

An example is given of a real series $\sum a_n$ with partial sums $s_n = a_0 + \cdots + a_n$ and power series transform $F(t) = \sum a_n t^n$ such that $na_n \geqq O(1)$, $F(t) \leqq O(1)$, and, for an infinite set of values of n, $s_n > \frac{1}{2}\log\log n$. It is then proved that if $na_n \geqq O(1)$ and $F(t) \leqq O(1)$, then $s_n \leqq O(\log\log n)$. There are several misprints.

R. P. Agnew (Ithaca, N. Y.).

Postnikov, A. G. The remainder term in the Tauberian theorem of Hardy and Littlewood. Doklady Akad. Nauk SSSR (N.S.) 77, 193–196 (1951). (Russian)

It is proved that, if $\sum_1^{\infty} a_n e^{-\lambda_n \sigma} = \sigma^{-1} + O(1)$ as $\sigma \to +0$, and if $a_n \geqq 0$, then $\sum_{\lambda_n \leqq P} a_n = P + O[P/\sqrt{(\log P)}]$ as $P \to \infty$. The idea is to consider $\sum a_n e^{-\lambda_n \sigma} f(e^{-\lambda_n \sigma})$ with a suitable $f(x)$ and to approximate to $f(x)$ in $[0, 1]$ by a polynomial $P_N(x) = \sum_0^N b_i x^i$, as in Karamata's method. But $P_N(x)$ is now chosen in a special way, namely as the best approximation to $f(x)$ in the sense of minimizing $E_N = \max_{0 \leqq x \leqq 1} |f(x) - P_N(x)|$ (for a given continuous $f(x)$ and given N). With this choice, the terms of $\sum |b_i|$ (and so the sum itself) do not exceed the corresponding expressions for the polynomial $2M \cos\{N \arccos(2x-1)\}$, where $M = \max_{0 \leqq x \leqq 1} |f(x)|$; while $E_N < 12\omega(1/2N)$, where $\omega(\delta)$ is the modulus of continuity of $f(x)$. This leads to

$$\sigma \sum_1^{\infty} a_n e^{-\lambda_n \sigma} f(e^{-\lambda_n \sigma}) = \int_0^1 f(x)dx + O(M 6^N \sigma) + O(\omega(1/2N)).$$

The stated result is obtained by taking $f(x)$ to be 0 in $[0, e^{-\alpha}]$, $1/x$ in $[e^{-\beta}, 1]$, linear in $[e^{-\alpha}, e^{-\beta}]$, and choosing $\sigma = 1/P$, $N = [c \log P]$ $(0 < c < 1/\log 6)$, $\alpha - \beta = 1/\sqrt{(\log P)}$, β or $\alpha = 1$ (to obtain alternative inequalities for $\sum_{\lambda_n \leqq P} a_n$).

A. E. Ingham (Cambridge, England).

Sunouchi, Gen-ichirô. On Mercer's theorem. Proc. Japan Acad. 22, no. 11, 360–361 (1946).

The author proves the following absolute convergence analogues of a generalized form of Mercer's theorem. (1) If $a_n > 0$, $\sum 1/a_n = \infty$ and $y_n = (1+a_n)t_n - a_n t_{n-1}$, then $\sum |\Delta y_n| < \infty$ implies $\sum |\Delta t_n| < \infty$. (2) If $q > -1$, $\lambda_n > 0$, $\sum \lambda_n/(\lambda_1 + \cdots + \lambda_{n-1}) = \infty$, and

$$y_n = x_n + q(\lambda_1 x_1 + \cdots + \lambda_n x_n)/(\lambda_1 + \cdots + \lambda_n),$$

then $\sum |\Delta x_n| < \infty$ implies $\sum |\Delta y_n| < \infty$. The second is a corollary of the first.

R. P. Boas, Jr. (Evanston, Ill.).

Lauwerier, H. A. The calculation of the coefficients of certain asymptotic series by means of linear recurrent relations. Appl. Sci. Research B. 2, 77–84 (1951).

The asymptotic expansion of an integral $\int_L e^{-\omega f(u)} g(u) du$ for large values of ω is required. It is assumed that the origin is a saddle point near which

$$f(u) = u^a \sum_0^{\infty} b_j u^j = u^a h(u) \qquad (b_0 > 0, a \geqq 2).$$

If we put $f(u) = v^a$ and take v as new variable, we have to find the asymptotic expansions of a number of integrals, each of the form

$$\int_0^{\infty \exp 2l\pi i/a} e^{-\omega v^a} g(u) \frac{du}{dv} dv,$$

and so the problem is reduced to that of finding the coefficients m_k in the power series $g(u)du/dv = \sum_0^{\infty} m_k v^k$. It is proved here that

$$m_k = \frac{1}{\Gamma((k+1)/a)} \int_0^{\infty} e^{-b_0 s} s^{(k+1)/a - 1} q_k(s) ds$$

where the polynomials $q_k(s)$, defined by

$$g(u) \exp[s\{b_0 - h(u)\}] = \sum_0^{\infty} q_j(s) u^j,$$

can be readily calculated from a recurrence relation. The method is applied here to $J_\nu(\omega)$, $\Gamma(\omega)$, $He_\nu(2\sqrt{\omega})$ and $(2\omega)^{\frac{1}{2}}M_{\frac{1}{2}\nu,0}(2\omega)$. *E. T. Copson* (St. Andrews).

Fourier Series and Generalizations, Integral Transforms

Sunouchi, Gen-ichirô. Trigonometrical interpolation. Proc. Japan Acad. **22**, no. 11, 362–365 (1946).

Let $f(x)$ be continuous with period 2π, $U_n(f, x)$ the trigonometric polynomial coinciding with $f(x)$ at the $2n+1$ points $2\pi k/(2n+1)$. Marcinkiewicz [Studia Math. 6, 1–17 (1936)] constructed an $f(x)$ not satisfying

$$\sum_{n=1}^{N}|U_n(f, x)-f(x)|=O(N),$$

$x\neq 0, 2\pi$. Let $\omega(h)$ be the modulus of continuity of $f(x)$. The author shows that if $\sum n^{-1}[\omega(n^{-1})]^p<\infty$, where $1\leqq p<\infty$, then $\sum n^{-1}|U_n(f, x)-f(x)|^p=o(n)$ almost everywhere; and if $\sum[\omega(n_k^{-1})]^p$ converges, then $U_{n_k}(f, x)$ converges to $f(x)$ almost everywhere. The paper also contains analogues of known results on Fourier series, generalizing results of Marcinkiewicz for $p=2$ [Studia Math. 6, 67–81 (1936); Mathematica, Cluj 14, 36–38 (1938)], from which it follows in particular that if $f'(x)\epsilon L^p$ and $f(0)=f(2\pi)=0$, and U_n' is the derivative of U_n, then $\sum_{n=1}^{N}|U_n'(f, x)-f'(x)|^2=o(N)$ almost everywhere and $U'_{n_k}\rightarrow f'(x)$ almost everywhere if $n_{k+1}/n_k>a>1$. *R. P. Boas, Jr.* (Evanston, Ill.).

Sunouchi, Gen-Ichirô, and Yano, Shigeki. Notes on Fourier analysis. XXX. On the absolute convergence of certain series of functions. Proc. Amer. Math. Soc. **2**, 380–389 (1951).

In the first section the authors extend some work of O. Szász [Ann. of Math. (2) 47, 213–220 (1946); these Rev. 7, 435]. A specimen result is as follows: If $\varphi(x)$ has period one and $0<d<\varphi(x)$ for $0\leqq a<b\leqq 1$, if $a_{n+1}>Ca_n>0$ and $\sum_{n=1}^{\infty}a_n\varphi(nx)$ is absolutely convergent at an irrational point, then $\sum_{n=1}^{\infty}a_n$ is convergent. The techniques of Mazur and Orlicz [Studia Math. 8, 1–16 (1940); these Rev. 3, 107] are used in the second portion of the paper to extend the theorem of Cantor and Lebesgue to fairly general series in two variables. *P. Civin* (Eugene, Ore.).

Sunouchi, Gen-Ichirô. On the Walsh-Kaczmarz series. Proc. Amer. Math. Soc. **2**, 5–11 (1951).

The functions $\psi_n(t)$ of Walsh and Kaczmarz complete the orthonormal system of Rademacher over the interval $\langle 0, 1\rangle$ [cf. Zygmund, Trigonometric Series, Warszawa-Lwów, 1935, pp. 5 and 13]. The Fourier series $f(t)\sim\sum c_n\psi_n(t)$ of an L-integrable function $f(t)$ with respect to the $\psi_n(t)$ is considered, and certain inequalities involving the partial sums $s_n(t)$ and their arithmetical means $\sigma_n(t)$ are proved. For instance, if f is L^r where $r>1$, then

$$A_r\int_0^1\left(\sum_{n=1}^{\infty}|s_{2^n}(t)-\sigma_{2^n}(t)|^2\right)^{r/2}dt$$
$$\leqq\int_0^1|f(t)|^r dt\leqq B_r\int_0^1\left(\sum_{n=1}^{\infty}|s_{2^n}(t)-\sigma_{2^n}(t)|^2\right)^{r/2}dt,$$

and $\int_0^1(\sup_n|\sigma_n(t)|)^r dt\leqq C_r\int_0^1|f(t)|^r dt$. The results are known in the trigonometric case [compare also papers by Paley, Proc. London Math. Soc. (2) 34, 241–264, 265–279 (1932); and Zygmund, Proc. Cambridge Philos. Soc. 34, 125–133 (1938)]. *W. W. Rogosinski* (Newcastle-upon-Tyne).

Lenz, Hanfried. Abschätzung einiger trigonometrischer Summen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1950**, 111–115 (1951).

A bound of $a_1/\sin(a_1/2A)$ is obtained for trigonometric polynomials of the type

$$\sum_{k=1}^{n}a_k\sin 2kx \text{ and } \sum_{k=1}^{n}a_k\sin(2k-1)x$$

where $0\leqq a_k\leqq A/k$ and $0\leqq\Delta a_k\leqq A/k^2$. For $a_k=1/k$ the bounds thus obtained are better than the previously known bounds. *P. Civin* (Eugene, Ore.).

Tsuchikura, Tamotsu. Notes on Fourier analysis (XXVI): Lipschitz condition of partial sums of Fourier series. Tôhoku Math. J. (2) **2**, 24–29 (1950).

Let $f(x)$ be continuous with period 2π and let $s_n(x)$ be the nth partial sum of its Fourier series. The main object of this paper is to prove that: (1) If $0<\alpha\leqq 1$ and if (*) $\max_x|s_n(x+\delta)-s_n(x)|=O(|\delta|^\alpha)$ uniformly in n, then $f(x)\epsilon\mathrm{Lip}\,\alpha$ and $|f-s_n|=O(n^{-\alpha})$ uniformly in x. (2) If $0<\alpha<1$ and $|f-s_n|=O(n^{-\alpha})$ uniformly in x, then $f\epsilon\mathrm{Lip}\,\alpha$ and (*) holds uniformly in n. (3) If $f\epsilon\mathrm{Lip}\,1$ and $|f-s_n|=O(n^{-1})$ then (*) holds uniformly in n with $\alpha=1$. Analogous results are also given for integrated Lipschitz conditions.
R. Salem (Cambridge, Mass.).

Civin, Paul. Approximation to conjugate functions. Proc. Amer. Math. Soc. **2**, 207–208 (1951).

The object of this paper is to prove that if $\{T_n(x)\}$ is a sequence of trigonometric polynomials of order n and if $|f(x)-T_n(x)|=O(n^{-\alpha})$, $\alpha>0$, uniformly in x, then the conjugate function $\tilde{f}$ and the conjugate polynomials $\tilde{T}_n$ satisfy $|\tilde{f}(x)-\tilde{T}_n(x)|=O(n^{-\alpha}\log n)$ uniformly in x, and that the order $n^{-\alpha}\log n$ cannot be improved. *R. Salem.*

Zamansky, Marc. Sur les théorème de Kuttner. C. R. Acad. Sci. Paris **232**, 2172–2174 (1951).

The author generalizes the well-known theorem of Kuttner [J. London Math. Soc. 10, 131–140 (1935)] asserting that if a trigonometric series S converges in a set E, and the conjugate series $\tilde{S}$ is summable $(C, 1)$ in E, then $\tilde{S}$ converges almost everywhere in E. The generalization is a special case of known results [see Plessner, C. R. (Doklady) Acad. Sci. URSS (N.S.) 9 (1935 IV), 251–253; Marcinkiewicz and Zygmund, Fund. Math. 26, 1–43 (1936); Trans. Amer. Math. Soc. 50, 407–453 (1941); these Rev. 3, 105].
A. Zygmund (Chicago, Ill.).

Ogievečkiĭ, I. I. On a precise estimate. Doklady Akad. Nauk SSSR (N.S.) **78**, 201–204 (1951). (Russian)

Let W^r be the class of functions of period 2π such that $|f^{(r)}(x)|\leqq 1$ for all x. Let $U_n(f, x)$ denote either the partial sums or the $(C, 1)$ means of the Fourier series of f. The numerical values of the constants $\sup_{f\epsilon W^r}\max_x|f(x)-U_n(f, x)|$ were determined by Kolmogoroff [Ann. of Math. (2) 36, 521–526 (1935)] and Nikolsky [Trav. Inst. Math. Stekloff 15 (1945); these Rev. 7, 435]. In the present paper the author obtains corresponding results for

$$U_n(f, x)=\tfrac{1}{2}a_0+\sum_{k=1}^{N_n}q_{nk}(a_k\cos kx+b_k\sin kx)$$

provided the multipliers q_{nk} $(0\leqq n<\infty, 0\leqq k\leqq N_n)$ satisfy

the conditions (a) $q_{n0}=1$, $0<q_{nk}<1$ $(k>0)$ and

$$\text{(b)} \qquad 1+\sum_{1}^{N_n} q_{nk}\cos kx \geqq 0$$

for all n and x. *A. Zygmund* (Chicago, Ill.).

Ogieveckiĭ, I. I. On some properties of sine series of positive continuous functions which are convex upwards. Doklady Akad. Nauk SSSR (N.S.) 78, 13–16 (1951). (Russian)

Let $f(x)$, $0<x<\pi$, be positive and concave, and let $s_n(x)$ and $\sigma_n(x)$ be the partial sums and the $(C, 1)$ means of the sine development $\sum b_n \sin nx$ of $f(x)$. It is well known [Koschmieder, Monatsh. Math. Phys. 39, 321–344 (1932); Fejér, Z. Angew. Math. Mech. 13, 80–88 (1933)] that $s_n(x)\geqq 0$ and $\sigma_n(x)\leqq f(x)$ for $0<x<\pi$. The author extends these results to linear means defined by a matrix. (1) If a finite row matrix $\{q_{nk}\}$, where $n=0, 1, \cdots$ and $0\leqq k\leqq N_n$, satisfies the conditions $q_{n0}=1$, $0<q_{nk}<1$, and

$$1+2\sum_{1}^{N_n} q_{nk}\cos kx \geqq 0,$$

then $U_n\{f, x\}=\sum_1^{N_n} q_{nk}b_k \sin kx \leqq f(x)$ for $0<x<\pi$. (2) If $q_{nk}>0$ and $q_{nk}>q_{n,k+1}$, then $U_n\{f, x\}>0$ for $0<x<\pi$.

A. Zygmund (Chicago, Ill.).

Mohanty, R. On the absolute Riesz summability of Fourier series and allied series. Proc. London Math. Soc. (2) 52, 295–320 (1951).

Let $0<\lambda_1<\lambda_2<\cdots$, $\lambda_n\to\infty$, and $\sum a_n$ be a given infinite series. If $C_r(w)=\sum_{\lambda_n\leqq w} a_n(1-\lambda_n/w)^r$ $(r\geqq 0)$ is of bounded variation in (A, ∞), A being a positive number, then $\sum a_n$ is said to be summable $|R, \lambda_n, r|$. The author applies this summability to the theory of Fourier series, in the case $\lambda_n=\exp(n/(\log n)^\alpha)$ $(\alpha>1)$, $\lambda_n=\exp(n^\alpha)$ $(0<\alpha<1)$ and

$$\lambda_n=\exp((\log n)^\alpha) \qquad (\alpha>0).$$

He proves that the Fourier series of $f(t)$, at $t=x$ is summable

$$|R, \exp((\log n)^\alpha), 1| \ (\alpha>0), \ |R, \exp((\log n)^2), 2|$$

or $|R, \exp(n/(\log n)^\beta), 1|$ $(\beta=1+1/\delta)$, according as

$$\phi(t)\log\log(k/t), \quad \phi_1(t)\log(k/t),$$

or $\phi(t)/t^\delta$ $(\delta>0)$ is of bounded variation in $(0, \pi)$, where k is some constant and

$$\phi(t)=\tfrac{1}{2}[f(x+t)+f(x-t)], \quad \phi_1(t)=t^{-1}\int_0^t \phi(u)du.$$

The author proves similar theorems concerning allied series, differentiated Fourier series, and differentiated allied series of $f(t)$. *S. Izumi* (Tokyo).

Matveev, I. V. On methods of summation of double Fourier series. Doklady Akad. Nauk SSSR (N.S.) 77, 957–960 (1951). (Russian)

Let c_{mn} be the Fourier coefficients of a function $f(x, y)$ of period 2π both with respect to x and y. Given a system of numbers $\lambda_{kl}^{(mn)}$ satisfying the conditions $\lambda_{-k,l}=\lambda_{k,-l}=\lambda_{k,l}$, the author introduces the linear means

$$U_{mn}(f; x, y)=\sum_{k=-m}^{m}\sum_{l=-n}^{n}\lambda_{kl}^{(mn)}e^{i(kx+ly)}$$

of the series $\sum c_{kl}e^{i(kx+ly)}$ and asks for conditions guaranteeing the uniform convergence of $U_{mn}(f; x, y)$ to $f(x, y)$, for every continuous f, as m and n tend to $+\infty$ independently of each other. He obtains a set of necessary and another set of sufficient conditions; and also a set of conditions both necessary and sufficient, provided the λ's display a rather regular behavior. The conditions involve differences of order two and four of the numbers λ_{kl} and are not simple enough to be reproduced here. *A. Zygmund.*

Gahariya, K. K. The summation of double trigonometric series by Riemann's method. Mat. Sbornik N.S. 28(70), 337–350 (1951). (Russian)

A double trigonometric series (1) $\sum_{-\infty,-\infty}^{+\infty,+\infty} c_{mn}e^{i(mx+ny)}$ is said to be Riemann summable to sum s, if the series

$$\text{(2)} \qquad \sum c_{mn}e^{i(mx+ny)}\left(\frac{\sin mu}{mu}\right)^2\left(\frac{\sin nv}{nv}\right)^2$$

converges for all values of u, v sufficiently small and different from zero, and if (2) tends to limit s as $u\to 0$, $v\to 0$. As in the case of series of one variable, Riemann summability can be associated with the existence of a certain generalized derivative of the function obtained by integrating (1) twice with respect to x and twice with respect to y. By taking $x=y=0$, we introduce Riemann summability of the numerical series $\sum c_{mn}$. The author shows that if all the symmetric partial sums of the latter series are bounded and if (3) $c_{mn}\to 0$ for $|m|+|n|\to\infty$, then ordinary [Pringsheim] convergence of $\sum c_{mn}$ implies Riemann summability of the latter series. The main result of the paper is the following one. Suppose that the series (1) is Riemann summable to a function $f(x, y)$ for all x, y, that condition (3) is satisfied, and that the symmetric partial sums $S_{mn}(x, y)$ of (1) satisfy an inequality (4) $|S_{mn}(x, y)|\leqq\varphi(x, y)$, $m, n=0, 1, \cdots$, where φ is finite and L-integrable over $0\leqq x\leqq 2\pi$, $0\leqq y\leqq 2\pi$. Then (1) is the Fourier series of $f(x, y)$. [Reviewer's remark: Condition (4) is so strong that it alone leads to the Fourier character of the series (1), without any application of the Riemann method, for it implies that

$$\int_0^{2\pi}\int_0^{2\pi}|S_{mn}|dxdy\leqq\text{const.},$$

so that (1) is certainly a Fourier-Stieltjes series. Hence the arithmetic means $\sigma_{mn}(x, y)$ of the S_{mn} tend almost everywhere to a finite limit $f(x, y)$ provided $m=n$ [see A. Zygmund, Amer. J. Math. 69, 836–850 (1947); these Rev. 9, 235]. Since the σ_{mn} also satisfy (4), we have

$$\int_0^{2\pi}\int_0^{2\pi}|f-\sigma_{mm}|dxdy\to 0,$$

and (1) is the Fourier series of f.] *A. Zygmund.*

Dzyadyk, V. K. On best approximation in the mean of periodic functions with singularities. Doklady Akad. Nauk SSSR (N.S.) 77, 949–952 (1951). (Russian)

The function $D^r(t)=\sum_{k=1}^{\infty}k^{-r}\cos(kt+\frac{1}{2}\pi r)$ plays an important role in the theory of best approximation of periodic functions by trigonometric polynomials [see, e.g., J. Favard, Bull. Sci. Math. (2) 61, 209–224, 243–256 (1937)]. Its $(r-1)$st derivative has a jump at $t=0$, and is otherwise continuous in $(0, 2\pi)$. The trigonometric polynomial T^*_{n-1} of order $n-1$ giving best approximation to D^r in the metric L_1 is the polynomial interpolating $D^r(t)$ at the zeros of $\cos(kt+\frac{1}{2}\pi r)$. Moreover (*) $\int_{-\pi}^{\pi}|D^r(t)-T^*_{n-1}(t)|dt=4K_rn^{-r}$, where K_r depends on r only [loc. cit.]. The author shows that if in the integral (*) we remove an ϵ-neighborhood of $t=0$, the resulting integral will be $O(1/\epsilon^2n^{r+1})$. This result is applied to the proof of the following theorem. Suppose that $f(t)$ is of period 2π, that the derivative $f^{(r-1)}(t)$ is absolutely continuous, that $f^{(r)}(t)$ is of bounded variation, and

that $f^{(r)}(t)=g(t)+h(t)$, where $g(t)$ is the function of jumps and $h(t)$ is absolutely continuous. Then for the best approximation $E_n(f)_L$ of f by trigonometric polynomials of order n, in the metric L, we have the formula

$$E_n(f)_L \simeq 4\pi^{-2}K_{r+1}n^{-(r+1)}\sum|A_k|$$

where A_k are all the jumps of $f^{(r)}$ (assuming $\sum|A_k|>0$).

A. Zygmund (Chicago, Ill.).

Redheffer, R. M. Remarks on incompleteness of $\{e^{i\lambda_n x}\}$, nonaveraging sets, and entire functions. Proc. Amer. Math. Soc. 2, 365–369 (1951).

Theorem 1: $\{e^{\pm i\lambda_n x}\}$ is not closed in an interval of length $2\pi d$, if $\limsup_{y\to\infty} \limsup_{x\to\infty} y^{-1}\sum_{x\leq\lambda_n<x+y}1<d$. The proof of this result uses a technique similar to that of Levinson [Gap and Density Theorems, Amer. Math. Soc. Colloquium Publ., v. 26, New York, 1940; these Rev. 2, 180]. The author's method gives simplified proofs of some results of Levinson [op. cit., theorems 30, 31]. A nonaveraging set of positive integers is defined as a set not containing the arithmetic mean of any two of its members. Theorem 2: A necessary and sufficient condition that all nonaveraging sets have density zero is that for each nonaveraging set $\{\lambda_n\}$, $\{e^{i\lambda_n x}\}$ is incomplete in every interval

W. H. J. Fuchs (Ithaca, N. Y.).

Lukacs, Eugene, and Szász, Otto. Certain Fourier transforms of distributions. Canadian J. Math. 3, 140–144 (1951).

Essentiellement, les auteurs démontrent que si l'inverse $1/P(t)$ d'un polynome $P(t)$ est une fonction caractéristique et si $t=b+ia$ est une racine de $P(t)$, (1) a ne peut pas être nul, (2) si $b\neq 0$, $-b+ia$ est aussi racine de $P(t)$, et il existe une racine imaginaire pure au moins $i\alpha$ de $P(t)$ telle que $\operatorname{sgn}\alpha=\operatorname{sgn}a$ et que $|\alpha|\leq|a|$. Ceci suppose que toutes les racines de $P(t)$ sont simples. *R. Fortet* (Caen).

Hirschman, I. I., Jr. The behavior at infinity of certain convolution transforms. Trans. Amer. Math. Soc. 70, 1–14 (1951).

Es seien b und a_k ($k=1, 2, \cdots$) reelle Konstanten, wobei

$$\sum_{k=1}^{\infty}a_k^{-2}<\infty,\quad E(s)=e^{bs}\prod_1^{\infty}(1-s/a_k)e^{s/a_k},$$

$$G(t)=(1/2\pi i)\int_{-i\infty}^{+i\infty}\{e^{st}/E(s)\}dt\quad(-\infty<t<+\infty),$$

$f(x)=\int_{-\infty}^{+\infty}G(x-t)d\alpha(t)$, und $\alpha(t)$ sei für alle endlichen Intervalle von beschränkter Variation. Der Verf. und Widder haben $G(t)$ und $f(x)$ in früheren Publikationen untersucht. Der Verf. benützt die folgenden Bezeichnungen: $G(t)\subset$Klasse Ia, wenn die a_k zum Teil positiv und zum Teil negativ und $\sum_{k=1}^{\infty}1/|a_k|=\infty$; $G(t)\subset$Klasse II, wenn alle a_k positiv und $\sum_{k=1}^{\infty}1/a_k=\infty$; $G(t)\subset$Klasse III, wenn die a_k positiv und $\sum_{k=1}^{\infty}1/a_k<\infty$. Der Verf. beweist den Satz: Wenn $f(x)=O[G(x)-\zeta]$, $x\to+\infty$, so ist $\alpha(t)$ für $t>\zeta$ konstant. Für Funktionen von der Klasse Ia und III gibt er analoge Sätze. *W. Saxer* (Zürich).

Carstoiu, Ion. Sur le logarithme intégral. C. R. Acad. Sci. Paris 232, 1624–1625 (1951).

Some remarks on the well-known occurrence of the exponential integral function in operational calculus.

A. Erdélyi (Pasadena, Calif.).

Carstoiu, Ion. Un nouveau théorème du produit dans le calcul symbolique. C. R. Acad. Sci. Paris 232, 1733–1734 (1951).

Formal derivation of the well-known complex convolution theorem of the Laplace transformation, with some applications. *A. Erdélyi* (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Marković, D. Domaines contenant le zéro du plus petit module des polynomes. Acad. Serbe Sci. Publ. Inst. Math. 3, 197–200 (1950).

For the polynomial $P(z)=1-z^p+a_{q+1}z^{q+1}+\cdots+a_nz^n$ with $1\leq p\leq q$, the author proves that the zero of smallest modulus lies in each of the domains $|z^k-1/p|\leq(n-1)/p$ provided $k=mp\leq q$ where m is a positive integer, and in the circle $|z^q|\leq(n-1)$ if p is not a factor of q. The proof follows from the observation that, if s_k denotes the sum of the reciprocal kth powers of the zeros of $P(z)$, then for $k\leq q$ $s_k=p$ or 0 according as k is or is not a multiple of p.

M. Marden (Milwaukee, Wis.).

Timan, A. F. A strengthening of Jackson's theorem on the best approximation of continuous functions by polynomials on a finite segment of the real axis. Doklady Akad. Nauk SSSR (N.S.) 78, 17–20 (1951). (Russian)

Let $f(x)$, $-1\leq x\leq+1$, have a continuous derivative $f^{(r)}(x)$, $r\geq 0$. A well-known theorem of Jackson [see, e.g., Theory of Approximation, Amer. Math. Soc. Colloq. Publ., v. 11, New York, 1930, p. 18] asserts that for every $n=1, 2, \cdots$ there is a polynomial $P_n(x)$ of degree at most n such that (*) $|f(x)-P_n(x)|\leq Cn^{-r}\omega_r(1/n)$, where ω_r is the modulus of continuity of f, and C depends on r only. The author shows that the right side of (*) can be replaced by $Cn^{-r}\{(1-x^2)^{\frac{1}{2}}+(|x|/n)^r\}\{\omega_r((1-x^2)^{\frac{1}{2}}/n)+\omega_r(|x|/n^2)\}$.

A. Zygmund (Chicago, Ill.).

Timan, A. F. The approximation of functions satisfying a Lipschitz condition by ordinary polynomials. Doklady Akad. Nauk SSSR (N.S.) 77, 969–972 (1951). (Russian)

Let $H^\alpha M$ denote the class of functions $f(x)$ defined for $-1\leq x\leq+1$ and satisfying there the condition

$$|f(x_1)-f(x_2)|\leq M|x_1-x_2|^\alpha.$$

Let $\hat{T}_k(x)=(2/\pi)^{\frac{1}{2}}\cos k \arccos x$ for $k=1, 2, \cdots$, and $\hat{T}_0(x)=(1/\pi)^{\frac{1}{2}}$. The system $\{\hat{T}_k\}$ is orthonormal over $(-1, +1)$, with respect to the weight function $(1-x^2)^{-\frac{1}{2}}$. Let $S_n(x, f)$ be the nth partial sum of the Fourier series of f, with respect to the system $\{\hat{T}_n\}$. Then for any $0<\alpha\leq 1$ the quantity $E_{S_n}^{(\alpha)}(x)=\sup_{f\epsilon H^\alpha M}|f(x)-S_n(f, x)|$ is

$$2^{\alpha+1}\pi^{-2}M(1-x^2)^{\frac{1}{2}\alpha}n^{-1}\log n\int_0^{\frac{1}{2}\pi}t^\alpha\sin t\,dt+O(n^{-\alpha}),$$

uniformly in $-1\leq x\leq+1$. An analogous relation is obtained for the arithmetic means $\sigma_n(f, x)$ of the $S_k(f, x)$. A special case of the latter relation is the inequality

$$|f(\pm 1)-\sigma_n(f, \pm 1)|\leq M/n,$$

which cannot be improved. Also, if $f\epsilon H^\alpha M$, then there is a constant C independent of x and n such that for every $n=1, 2, \cdots$ there is a polynomial $P_n(x)$ of degree at most n and satisfying $|f(x)-P_n(x)|\leq CMn^{-\alpha}[(1-x^2)^{\frac{1}{2}\alpha}+(|x|/n)^\alpha]$. All the results are stated without proof. *A. Zygmund*.

Cenov, I. V. On a question of the approximation of functions by polynomials. Mat. Sbornik N.S. **28**(70), 473–478 (1951). (Russian)

The author proves the following theorems on the polynomial $P(x)$ of degree at most n which minimizes (1) $\int|f(x)-P(x)|^s dx$, where $f(x)$ is continuous, $s\geqq 1$, and the integral is over (a, b). (I) Let f and ϕ have $(n+1)$th derivatives. Let $f_n(x)$, $\phi_n(x)$ be Lagrange $(n+1)$-point interpolation polynomials for f and ϕ, and let $R_n=f-f_n$, $S_n=\phi-\phi_n$; then if (2) $|f^{(n+1)}(x)|<|\phi^{(n+1)}(x)|$ it follows that $|R_n(x)|<|S_n(x)|$. (II) If $\rho_s(f)$ is the minimum of (1), $P^*(x)$ is the corresponding minimizing polynomial, and $Q^*(x)$ is the corresponding minimizing polynomial for ϕ, and if each of the equations $f(x)=P^*(x)$ and $\phi(x)=Q^*(x)$ has at least $n+1$ distinct roots, then (2) implies (3) $\rho_s(f)<\rho_s(\phi)$. (III) The equation $f(x)=P^*(x)$ has at least $n+1$ roots. (IV) Hence (2) implies (3). [For the author's earlier results in this direction see same Sbornik N.S. **21**(63), 435–438 (1947); these Rev. **9**, 282.] *R. P. Boas, Jr.*

Gukevič, V. I. The best approximation in the mean of the function $\ln(a-x)$ **by polynomials.** Doklady Akad. Nauk SSSR (N.S.) **77**, 785–786 (1951). (Russian)

The author calculates the quantity described in the title for the interval $(-1, 1)$ and $-1<a<1$; for polynomials of degree $2n-1$ it is $c(1-a^2)^{\frac{1}{2}}n^{-1}+O(n^{-2}\log n)$, where c is given explicitly as a definite integral. The corresponding result for $|a|\geqq 1$ was known. *R. P. Boas, Jr.*

Special Functions

Schmid, Hermann Ludwig. Über Polynomkettenbrüche. Math. Nachr. **4**, 481–488 (1951).

For given polynomials $P(x)$, $Q(x)$, and for a particular integral $H(x)$ of the hypergeometric differential equation the author shows that the function $z=PH+QH'$ satisfies the differential equations

$$(1) \qquad A_n z^{(n)}=B_n z^{(n+1)}+C_n z^{(n+2)} \quad (n=0, 1, 2, \cdots),$$

where the coefficients A_n, B_n, C_n are polynomials in x and n. Formulas for these coefficients are given and bounds for the degree of the coefficients, in both x and n, are obtained. Under suitable conditions it is shown that the continued fraction expansion for A_0z/z' determined by the system (1) converges to A_0z/z'. *W. T. Scott* (Evanston, Ill.).

Buchholz, Herbert. Die Summe der reziproken Potenzen der Nullstellen von $M_{\kappa,\mu/2}(z)$ **hinsichtlich** z**.** Z. Angew. Math. Mech. **31**, 149–152 (1951).

The author discusses the zeros $z=a_1, a_2, \cdots$ of

$${}_1F_1(\tfrac{1}{2}(1+\mu)-\kappa; 1+\mu; z),$$

sums the infinite series $\sum_{n=1}^{\infty} z/a_n(z-a_n)$, writes down explicit expressions for $S_p=\sum a_n^{-p}$ when $p=2, 3, 4, 5, 6$, and describes various properties of S_p for general (positive integer) p. *A. Erdélyi* (Pasadena, Calif.).

Bouwkamp, C. J., and Bremmer, H. A note on Kline's Bessel-function expansion. Nederl. Akad. Wetensch. Proc. Ser. A. **54**=Indagationes Math. **13**, 130–134 (1951).

The function

$$G_\nu(\theta, z)=J_\nu(\nu\sec\theta)Y_\nu(z+\nu\sec\theta)-Y_\nu(\nu\sec\theta)J_\nu(z+\nu\sec\theta)$$

has been expanded in the form

$$G_\nu(\theta, z)=\pi^{-1}\sum_{n=1}^{\infty}a_n(\theta, z)(\nu^{-1}\cos\theta)^n$$

by M. Kline [Proc. Amer. Math. Soc. **1**, 543–552 (1950); these Rev. **12**, 334]. The present paper contains a new proof of this expansion. The proof is based on the differential equation satisfied by G_ν as a function of z, and leads to a recurrence relation for the coefficients $a_n(\theta, z)$. *A. Erdélyi* (Pasadena, Calif.).

Pinney, Edmund. A theorem of use in wave theory. J. Math. Physics **30**, 1–10 (1951).

Das vom Verf. ins Auge gefasste Theorem bezieht sich auf die Auswertung eines unendlichen Integrals über ein Konturintegral, dessen Integrand das Produkt einer Besselscher Funktion erster Art, nullter Ordnung und einiger Exponentialfunktionen enthält. Zum Beweis dieses Theorems beweist Verf. zunächst zwei Sätze, welche sich auf etwas einfachere analoge Doppelintegrale beziehen. Mit Hilfe dieser beiden Sätze gelingt es dann, das gewünschte Theorem zu beweisen. Zum Schluss gibt Verf. eine Anwendung des betreffenden Theorems auf die Theorie der elastischen Wellen. *M. J. O. Strutt* (Zürich).

Turán, Paul. On the zeros of the polynomials of Legendre. Časopis Pěst. Mat. Fys. **75**, 113–122 (1950). (English. Czech summary)

Let $1>x_{1n}>x_{1,n-1}>\cdots>x_{nn}>-1$ be the zeros of the nth Legendre polynomial $P_n(x)$. The author gives his original proof (said to have been found in 1941 and written to Szegő in 1946) of his theorem that

$$x_{1n}-x_{1,n-1}<x_{2n}-x_{2,n-1}<\cdots<x_{mn}-x_{m,n-1},$$

where $m=[\frac{1}{2}(n-1)]$. The proof depends on the author's lemma that $\Delta_n(x)=P_n^2-P_{n-1}P_{n+1}>0$ for $-1<x<1$. The latter is proved by finding the intervals where Δ_n and Δ_n' are positive definite quadratic forms in P_n, P_{n-1} and in P_n', P'_{n-1}, respectively. For many other proofs and interesting generalizations of Turán's inequality, see G. Szegő [Bull. Amer. Math. Soc. **54**, 401–405 (1948); these Rev. **9**, 429], Beckenbach, Seidel, and Szász [Duke Math. J. **18**, 1–10 (1951); these Rev. **12**, 702], and the references cited in the latter paper. As an unexpected application of his theorem, the author outlines a future paper with Fejér, in which necessary and sufficient conditions are given for certain Riemann sums of certain functions $f(t)$ to converge monotonically to $\int_0^1 f(t)dt$. *G. E. Forsythe.*

Unger, Heinz. Lommelsche Polynome und Ableitungspolynome bei der numerischen Berechnung von Zylinderfunktionen. Arch. Math. **2**, 375–381 (1950).

Let $Z_\nu(z)$ be any Bessel function of order ν and variable z. Given Z_ν and $Z_{\nu-1}$ (or $Z_{\nu+1}$), the function $Z_{\nu+n}$ (n integer) may be computed by a repeated application of the recurrence relations. The process can be condensed into one step,

$$(1) \qquad Z_{\nu+n}(z)=Z_\nu(z)R_{\nu,n}(z)-Z_{\nu-1}(z)R_{\nu+1,n-1}(z)$$

where the $R(z)$ are Lommel's polynomials. The author recommends computation from (1) up to $n=10$ or 12. For larger values of n it is better to break up the process in two or three steps. He gives a similar discussion of the pth derivatives, with respect to z, of $Z_\nu(z)$ and $Z_{\nu+n}(z)$. *A. Erdélyi* (Pasadena, Calif.).

Luke, Yudell L., and Ufford, Dolores. Concerning a definite integral. J. Aeronaut. Sci. **18**, 429–430 (1951).

Fettis [same J. **17**, 184–185 (1950); these Rev. **11**, 594] expressed the integral $\int_0^\infty e^{-it}(zt)^{-1}[z+t-(z^2+t^2)^{\frac{1}{2}}]dt$ in

terms of tabulated functions, except for one part which was expressed as a power series. In the present note the integral is expressed completely in terms of tabulated functions.

A. Erdélyi (Pasadena, Calif.).

Kreyszig, E. Über den allgemeinen Integralsinus Si (z, α). Acta Math. **85**, 117–181 (1951).

The properties of a function defined by

$$\mathrm{Si}(z, \mu) = \int_0^z t^{-\mu} \sin t dt, \qquad 0 < \Re\mu < 2,$$

which in case $\mu = 1$ reduces to the well-known sine integral $\mathrm{Si}(z)$, are investigated. The real and imaginary parts of μ are supposed to be α and zero respectively, with $0<\alpha<2$. Various other expressions for $\mathrm{Si}(z, \alpha)$ (in the form of a Taylor series, expressed as an incomplete Gamma function, and as a Whittaker function) and an asymptotic expansion are derived. For $\alpha = 0$, $\mathrm{Si}(z, \alpha) = 1 - \cos z$, and the (double) zeros are $z_n = \pm 2\pi n$ $(n = 0, 1, 2, \cdots)$. For the zeros in the case $0<\alpha<2$ the following statements are made: (1) All the zeros are simple and conjugate complex with nonvanishing real or imaginary parts. (2) If α increases monotonously, starting with $\alpha = 0$, and tends to $\alpha = 2$, the real double zeros at $2\pi n$ disintegrate into two conjugate complex simple zeros whose real part tends monotonously to $(2n-1)\pi$ and whose imaginary parts tend monotonously to plus or minus infinity respectively. (3) There do not exist any zeros within the strip $2\pi n \leqq \Re(z) \leqq (2n+1)\pi$, and in the strip $0 < \Re(z) \leqq 2\pi n$ there exist $2n$ zeros. (4) The sequence $\sum_{n=1}^\infty |z_n|^{-1}$ is divergent, but $\sum_1^\infty |z_n|^{-1-\epsilon}$ is convergent when $\epsilon > 0$. Finally, the expression of $\mathrm{Si}(z, \alpha)$ as an infinite product and some hints for the numerical computation of the zeros are given. The paper concludes with numerical tables (12 pages) and diagrams (8 pages).

F. Oberhettinger (Washington, D. C.).

Harmonic Functions, Potential Theory

Rudin, Walter. A theorem on subharmonic functions. Proc. Amer. Math. Soc. **2**, 209–212 (1951).

L'auteur perfectionne dans le plan un critère de sousharmonicité pour en faire une petite application. Notons $\mathfrak{M}_r f(M)$ la moyenne de f sur la circonférence de centre M et rayon r, $\Delta_r f(M)$ la différence $\mathfrak{M}_r f(M) - f(M)$, enfin $\Lambda^* f(M)$ la lim $\sup_{r\to 0} (4/r^2)\Delta_r f$. Supposons f semicontinue supérieurement dans D ouvert et $\Lambda^* f$ existante (ce qui implique au sens de l'auteur la sommabilité de f sur les circonférences de rayon assez petit autour de chaque point). Alors on savait que la condition $\Lambda^* f \geqq 0$ entraine la sousharmonicité. L'auteur démontre ce résultat en supposant seulement $\Lambda^* f \geqq 0$ sauf sur un ensemble E dénombrable où lim $\sup_{r\to 0} \Delta_r f/r \geqq 0$. Je ferai remarquer qu'il suffit de la condition moins forte: $\Lambda^* f \geqq 0$ sauf sur E polaire où lim $\sup_{r\to 0} \Delta_r f \geqq 0$. Car f est alors $<+\infty$ donc bornée supérieurement localement, et $w = f + \epsilon v$ $(\epsilon > 0)$ où v est localement sousharmonique $\leqq 0$, infinie sur E, est sousharmonique et converge pour $\epsilon \to 0$ vers une fonction quasi-sousharmonique W_0; la régularisée de W_0, sousharmonique, vaut la $\lim_{r\to 0} \mathfrak{M}_r f$ nécessairement existante et égale à f (et cela s'étend aux espaces supérieurs).

M. Brelot.

Inoue, Masao. On the growth of subharmonic functions and its applications to a study of the minimum modulus of integral functions. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **1**, 71–82 (1950).

Au moyen d'une majoration de Beurling [thèse, Upsala, 1933], l'auteur, considérant une fonction sousharmonique dans un domaine plan infini, soumise à une majoration d'allure à l'infini, améliore cette majoration par l'hypothèse d'une autre majoration à la frontière et une hypothèse métrique sur le complémentaire. Étude d'un problème de Dirichlet très particulier pour domaine infini (restriction sur l'allure à l'infini de la solution). Applications aux fonctions entières.

M. Brelot (Grenoble).

Nehari, Zeev. Sur la conjuguée d'une fonction harmonique bornée. C. R. Acad. Sci. Paris **232**, 1626–1627 (1951).

Let $u(z)$ be harmonic and bounded, $|u(z)|<1$, in a multiply-connected domain D whose boundary consists of n closed analytic curves $C_1 \cdots, C_n$, and let $v(z)$, the conjugate of $u(z)$, be uniform in D. It is proved that, if x and y are different points of D, then

$$(1)\, |v(x)-v(y)| \leqq \frac{1}{\pi} \sum_{k=1}^{n} \left| V_k\Big[N(z, x) - N(z, y) + \sum_{\mu=1}^{n-1} f\lambda_\mu \omega_\mu^*(z)\Big]\right|,$$

where $N(z, x)$ is the Neumann function of D, $\omega_\mu^*(z)$ is the conjugate harmonic function of the harmonic measure $\omega_\mu(z)$ of D relative to C_μ, $V_k[\ ]$ is the maximal variation of the expression in brackets for z belonging to the curve C_k, and the λ_μ are arbitrary real numbers. The function $u(z) = U(z)$ for which the equality sign prevails in (1) is of the form $U(z) = \Re\{F(z)\}$ where $w = F(z)$ maps D conformally onto a surface consisting of n carbon copies of the infinite strip $-1 < \Re\{w\} < 1$. The result generalizes a classical inequality of H. A. Schwarz concerning the conjugate $v(z)$ of a function $u(z)$, harmonic and bounded, $|u(z)|<1$, in $|z|<1$: $|v(z)-v(0)| \leqq (2/\pi) \log\{(1+|z|)/(1-|z|)\}$.

A. J. Lohwater (Ann Arbor, Mich.).

Dinghas, Alexandre. Sur quelques théorèmes du type de Phragmèn-Lindelöf dans la théorie des fonctions harmoniques de plusieurs variables. C. R. Acad. Sci. Paris **232**, 1394–1395 (1951).

The author considers in an infinite region G of n-dimensional space bounded by a surface F passing through the origin the class of harmonic functions U which are nonpositive on F, but which are positive somewhere in G. If $M(r)$ denotes the maximum of U on the portion of the sphere of radius r about the origin which lies in G and if $\alpha > 1$, it is shown that $M(\alpha r) \geqq C_0 e^{\varphi_n(r)} + C_1$ for suitable constants C_1 and $C_0 > 0$, where $\varphi_n(r) = \int_{r_0}^r [a_n(x)/x] dx$, with a suitable positive r_0, and a suitable positive function $a_n(x)$. Proofs are based on Green's theorem and Schwarz's inequality.

P. R. Garabedian (Stanford University, Calif.).

Cheng, Min-Teh. On a theorem of Nicolesco and generalized Laplace operators. Proc. Amer. Math. Soc. **2**, 77–86 (1951).

L'auteur reprend dans l'espace à n dimensions un critère de polyharmonicité de Nicolesco généralisant celui de Blaschke pour l'harmonicité. Il montre par un exemple que la seule continuité ne suffit pas avec la condition-limite à base de moyennes et rappelle que le critère devient exact avec une certaine uniformité dans cette limite. Pour une fonction à dérivées secondes continues, la biharmonicité résulte de la nullité d'un certain nouveau laplacien généralisé;

dans le plan, on montre qu'elle résulte aussi d'une autre condition que Cioranesco avait donnée pour une fonction analytique mais qui est valable s'il y a des dérivées troisièmes continues. *M. Brelot* (Grenoble).

Ancora, Rosa Bianca. Problemi analitici connessi alla teoria della piastra elastica appoggiata. Rend. Sem. Mat. Univ. Padova **20**, 99–134 (1951).

On sait que le problème d'équilibre de la plaque encastrée revient à déterminer une fonction biharmonique u dans un domaine de telle sorte que u et Δu prennent des valeurs données (assez régulières) sur la frontière Γ (assez régulière). L'auteur traite la question avec Γ (de régularité peu précisée) pouvant avoir des points anguleux, en s'inspirant de méthodes de Picone et Fichera. Si ω_k est une suite de polynomes homogènes biharmoniques, linéairement indépendants et dont les combinaisons linéaires fournissent tous les polynomes biharmoniques, on démontre d'abord comme lemme que le système vectoriel $(\omega_k, \Delta\omega_k)$ possède la "complétion hilbertienne" sur Γ, c'est à dire que les conditions $\int_\Gamma(\varphi_1\omega_k+\varphi_2\Delta\omega_k)ds=0$ entrainent la nullité presque partout de φ_1 et φ_2 supposées "quasi continues" et du carré sommable sur Γ. Deux méthodes de résolution sont données à partir d'une suite de combinaisons linéaires de ω_k (avec étude de l'approximation) ou de certains potentiels de simple ou double couche. Des applications numériques sont développées. *M. Brelot* (Grenoble).

Ghizzetti, Aldo. Sui problemi di Dirichlet e di Neumann per l'ellisse. Rend. Sem. Mat. Univ. Padova **20**, 244–248 (1951).

Let U be the solution of the Dirichlet problem for the interior of an ellipse. The ellipse, slit along the segment joining the foci, can be mapped conformally on a circular annulus by an elementary function. The value of U is thus known on the outer perimeter of the annulus but not on the inner; however on the inner perimeter U must be an even function of θ and the normal derivative an odd function. If a Fourier series solution for U is assumed, the above conditions serve to determine explicity the Fourier coefficients and thus an explicit solution to the Dirichlet problem in the ellipse is obtained. The Neumann problem is treated similarly. *J. W. Green* (Princeton, N. J.).

Reynolds, R. R. The Dirichlet problem for multiply connected domains. J. Math. Physics **30**, 11–22 (1951).

The author has adapted Bergman's method of orthogonalizing complete sets of harmonic functions for use in numerical work. The essentials of this method and the basic formulas may be found in S. Bergman, The Kernel Function and Conformal Mapping, chapters 1, 5 [Amer. Math. Soc., New York, 1950; these Rev. **12**, 402]. The orthogonalization process has been reduced to a convenient schedule, and for the case of a domain consisting of a circle with a number of circular holes punched out, tables have been presented to facilitate the computation in closed form of the pertinent integrals. Also discussed are methods for estimating the error committed by stopping at the nth approximation $u^{(n)}(x, y)$. The following numerical example for a triply connected domain is presented. Let B designate the domain bounded by b_0: $|\zeta|=4$; b_1: $|\zeta+2|=1$; b_2: $|\zeta-2|=1$. The author obtains the seven termed approximant to the function which is harmonic in B, vanishes on b_0 and b_2, and is unity on b_1. *P. Davis* (New London, Conn.).

Power, G. Change in potential due to a dielectric sphere. Amer. Math. Monthly **58**, 249–253 (1951).

When a sphere of radius a, filled with homogeneous dielectric of specific inductive capacity k, is placed with its centre at the origin in any electrostatic field whose potential function is $\varphi(x, y, z)$, having no singularities inside or on the sphere, the potential function φ_1 inside the sphere and the potential function φ_2 outside are given by

$$\varphi_1=\frac{2}{k+1}\varphi(x, y, z)+\frac{k-1}{(k+1)^2}\int_0^1 t^{-k/(k+1)}\varphi(xt, yt, zt)dt,$$

$$\varphi_2=\varphi(x, y, z)-\frac{k-1}{k+1}\frac{a}{r}\varphi(x_1, y_1, z_1)+\frac{k-1}{k+1}\frac{a}{r}\int_0^1 t^{-k/(k+1)}\varphi(x_1t, y_1t, z_1t)dt,$$

where $x_1=a^2x/r^2$, $y_1=a^2y/r^2$, $z_1=a^2z/r^2$, $r^2=x^2+y^2+z^2$. It is assumed that no other boundaries are present. Putting $k=0$ in the expression for φ_2 yields the velocity potential for the flow past a sphere of incompressible inviscid fluid in terms of the undisturbed potential $\varphi(x, y, z)$, a theorem proved by P. Weiss [Proc. Cambridge Philos. Soc. **40**, 259–261 (1944); these Rev. **6**, 191]. *L. M. Milne-Thomson.*

Power, G., and Martin, A. I. The sphere theorem in hydrodynamics. Math. Gaz. **35**, 116–117 (1951).

Proof of a theorem of P. Weiss [Proc. Cambridge Philos. Soc. **40**, 259–261 (1944); these Rev. **6**, 191].

Inoue, Masao. On the resolution of $\Delta U=cU+\varphi$ by the iteration of averaging process. J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **1**, 83–91 (1950).

Given a function f, continuous on the boundary of a bounded domain D of three-dimensional space, let F be any continuous extension of f onto D. The author forms an infinite sequence of functions U_n (with $U_0=F$) by an iterative process involving averaging with respect to volume over spheres of nonincreasing radii. His principal theorem asserts that under certain conditions this sequence converges, uniformly on any closed subset of D, to the solution of the equation $\Delta U=cU+\varphi$ which takes on the boundary values f; here $c\geqq 0$ and φ are assumed to be continuous, together with their first partial derivatives, in D. The theorem also asserts that, without the differentiability condition on c and φ, a similar result holds with ΔU replaced by a certain generalized Laplacian. The author also states (without proof) that similar results may be obtained using peripheral means, and also circumferential or areal means in the plane. *F. W. Perkins* (Hanover, N. H.).

Haviland, E. K. A note on unrestricted solutions of the differential equation $\Delta u=f(u)$. J. London Math. Soc. **26**, 210–214 (1951).

Given the single-valued function $f(\zeta)$ in $C^{(2)}$ $(-\infty<\zeta<\infty)$ and such that $f(\zeta)\geqq c>0$ and $f''(\zeta)>0$, the author considers the differential equation (1) $\Delta u=f(u)$, in 3-space. By an unrestricted solution is meant a solution valid throughout the entire 3-space. The principal result established is contained in the following theorem: The convergence of $\int^\infty[F(\eta)]^{-\frac{1}{2}}d\eta$, where $F(\eta)=\int_0^\eta f(\zeta)d\zeta$, is necessary and sufficient for the nonexistence of unrestricted solutions of (1). The author notes that this corresponds to a result given by Wittich for the plane [Math. Z. **49**, 579–582 (1944); these Rev. **6**, 228]. *F. W. Perkins* (Hanover, N. H.).

Differential Equations

Kumorovitz, Michal. Une solution du système linéaire homogène d'équations différentielles du premier ordre à coefficients constants. Ann. Soc. Polon. Math. 23, 190–200 (1950).

The theory of linear first order systems of differential equations with constant coefficients is developed without making use of the canonical form of a square matrix. The general solution of the differential system is given explicitly in finite terms. The main tool is a theorem concerning the rank of the powers of a matrix, for which E. Weyr [On the Theory of Bilinear Forms, Prague, 1889, pp. 30–34] gave a proof that does not depend on the theory of elementary divisors. *W. Wasow* (Los Angeles, Calif.).

Levi, Eugenio. Sul comportamento asintotico delle soluzioni dei sistemi di equazioni differenziali lineari omogenee. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 9, 26–31 (1950).

This is part II of the result announced earlier [same Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 465–470 (1950); these Rev. 12, 611] and here the proof is completed.
N. Levinson (Cambridge, Mass.).

Rapoport, I. M. On linear differential equations with periodic coefficients. Doklady Akad. Nauk SSSR (N.S.) 76, 793–795 (1951). (Russian)

The author considers the question of stability of oscillations determined by the equations: (1) $dx_1/dt=\lambda x_2$, $dx_2/dt=-\lambda p(t)x_1$, $p(t+T)=p(t)$, $p(t)>0$, where $p(t)$ has a summable derivative of order $(m+1)$ in the interval $(0, T)$. Using the method of a previous paper [same Doklady N.S. 73, 889–890 (1950); these Rev. 12, 183] he constructs a solution of (1) and discusses the problem of stability, giving an asymptotic formula for the length of the zone of instability.
C. G. Maple (Washington, D. C.).

Amerio, Luigi. Sull'estensione delle nozioni di "colle", "nodo" e "fuoco" ai sistemi di due equazioni differenziali periodiche in tre variabili. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 10, 206–212 (1951).

Let $X(t, x, y)$, $Y(t, x, y)$ be functions which are analytic with respect to x and y and periodic with respect to t, with the common period T. The author considers the system of differential equations $dx/dt=X(t, x, y)$, $dy/dt=Y(t, x, y)$, and develops certain analogies with the Poincaré theory of singular points of a system of equations of the form: $dx/dt=X(x, y)$, $dy/dt=Y(x, y)$. In the case of the second system, a point $(\bar{x}, \bar{y})$ is a singular point if $X(\bar{x}, \bar{y})=Y(\bar{x}, \bar{y})=0$. To such a singular point there corresponds the (here called) singular solution $x=\bar{x}$, $y=\bar{y}$. In the case of the first system of equations, the author takes, as the analogue of such a singular solution, a solution in which x and y are periodic with the period T. It is shown that in the ordinary cases these periodic solutions can be classified in three types, which present definite analogies with the nodes, foci, and saddle points occurring in the Poincaré theory. Most of the paper is concerned with the analytical aspects of the analogies, the geometrical aspects being discussed only briefly. *L. A. MacColl* (New York, N. Y.).

Malkin, I. G. A theorem on stability in the first approximation. Doklady Akad. Nauk SSSR (N.S.) 76, 783–784 (1951). (Russian)

Let

$$\dot{x}_s=X_s^{(m)}(x_1, \cdots, x_n)+\varphi_s(t, x_1, \cdots, x_n), \tag{1}$$

$s=1, 2, \cdots, n$, where $X_s^{(m)}$ are forms of the mth degree ($m\geqq 1$) and $|\varphi_s|\leqq A(|x_1|+\cdots+|x_n|)^m$, A constant. The following theorem is proved: If $x=0$ is an asymptotically stable solution of the system $\dot{x}_s=X_s^{(m)}$, then the same is true for (1), whatever the functions φ_s may be, provided A is small enough. *J. L. Massera* (Montevideo).

Friedlander, F. G. On the recurrent solutions of a class of non-linear differential equations. Proc. Cambridge Philos. Soc. 47, 315–330 (1951).

In the system of differential equations

$$dx/dt=\omega y+kf(x, y, t), \quad dy/dt=-\omega x+kg(x, y, t)$$

let ω be a positive constant and k a small parameter. The functions f, g possess continuous partial derivatives of the second order satisfying Lipschitz conditions, and are periodic with period 2π in t. A solution of this system is called a simple recurrent solution, if it is either periodic with period $2n\pi$ or an almost periodic function of the type $x=H(\sigma t, t)$, $y=K(\sigma t, t)$, where σ is irrational and $H(u, v)$, $K(u, v)$ have least periods 2π in u and $2n\pi$ in v. The author proves several theorems concerning the existence and stability of such recurrent solutions. The most important one of these states that if ω is irrational and k sufficiently small, and if a certain auxiliary equation $F_0(r)=0$ has a simple zero $r=c$, then the differential system possesses, for small $|k|$, simple recurrent solutions for which $\lim_{k\to 0}(x^2+y^2)^{\frac{1}{2}}=c$, and any other solution whose initial point is sufficiently close to the circle $r=c$ tends to one of these recurrent solutions as t approaches infinity. In other theorems it is assumed that f and g are polynomials in x, y, $\cos t$, $\sin t$, but ω is permitted to approach a rational "subresonance" requency, as $k\to 0$.
W. Wasow (Los Angeles, Calif.).

Reuter, G. E. H. A boundedness theorem for non-linear differential equations of the second order. Proc. Cambridge Philos. Soc. 47, 49–54 (1951).

The differential equation $\ddot{x}+kf(x)\dot{x}+g(x)=kp(t)$, $k>0$, is considered where t is the independent variable. Results of Cartwright and Littlewood [Ann. of Math. (2) 48, 472–494 (1947); these Rev. 9, 35], and Newman [Compositio Math. 8, 142–156 (1950); these Rev. 12, 611] are improved on. It is assumed $f(x)$ and $g(x)$ are continuous and $p(t)$ is summable; $P(t)=\int_0^t p(t)dt$; $F(x)=\int_0^x f(x)dx$; and $G(x)$ is similarly related to $g(x)$. If $P(t)$ is bounded for $t\geqq 0$, $F(x)\to+\infty(-\infty)$ as $x\to+\infty(-\infty)$, $xg(x)>0$ for $|x|\geqq x_0$, and $G(x)\to+\infty$ as $|x|\to\infty$, then there exists a constant B depending only on properties of $f(x)$, $g(x)$, and $p(t)$ such that every solution ultimately satisfies $|x|<B$, $|\dot{x}|<B(k+1)$. The method of proof, the author observes, follows that of the reviewer [J. Math. Physics 22, 41–48 (1943); these Rev. 5, 66].
N. Levinson (Cambridge, Mass.).

Mikusiński, J. G.-. On Fite's oscillation theorems. Colloquium Math. 2, 34–39 (1949).

The author generalizes certain results of Fite [Trans. Amer. Math. Soc. 19, 341–352 (1918)] for the differential equation $y^{(n)}+A(x)y=0$. Here $A(x)$ is continuous if $x\geqq a$. If n is odd, $A(x)\geqq 0$, and $\int_a^\infty x^{n-1}A(x)dx=\infty$, then he proves each solution either oscillates or tends to zero monotonely as $x\to\infty$. If n is even $A(x)\geqq 0$, and $\int_a^\infty x^{n-1-\epsilon}A(x)dx=\infty$ for some $\epsilon>0$, then each solution oscillates as $x\to\infty$. The proof follows Fite.
N. Levinson (Cambridge, Mass.).

Pedrazzini, Pierino. Sulla risoluzione del sistema di equazioni differenziali $1+y'^2=s'^2$

$$x^2+y^2+\frac{2s}{s'}(x+yy')=m^2.$$

Period. Mat. (4) **29**, 42–44 (1951).

The general solution of the system of equations of the title is obtained in the form $x=\varphi(s, c_1, c_2)$, $y=\psi(s, c_1, c_2)$, the functions $\varphi(s, c_1, c_2)$, $\psi(s, c_1, c_2)$ being expressed explicitly in terms of elementary functions and an elliptic integral. *L. A. MacColl* (New York, N. Y.).

Conte, Luigi. A proposito di un'equazione differenziale del secondo ordine. Bull. Soc. Math. Phys. Macédoine **2**, 35–37 (1951).

The general solution of the equation

$$y''+A(x)y'+B(x)y=0,$$

in which $A(x)=-\lambda(d/dx)\log a(x)$,

$$B(x)=-(d^2/dx^2)\log a(x)+(\lambda-1)[(d/dx)\log a(x)]^2$$

(λ, a constant), is

$$y=\exp\left[-\lambda^{-1}\int A(x)dx\right] \times\left\{C_2+C_1\int\exp\left[(2-\lambda)\lambda^{-1}\int A(x)dx\right]dx\right\}.$$

Several equations in mathematical physics occur as special cases of the above. The author considers an analogous case of a third order differential equation. *W. J. Trjitzinsky.*

Mitrinovitch, Dragoslav S. Sur une équation différentielle indéterminée du second ordre. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 227–228 (1951).

In the membrane theory of shells of revolution, Neményi and Truesdell encountered the equation

$$(*)\qquad F''/F+m^2f''/f=0$$

[cf. Truesdell, Trans. Amer. Math. Soc. **58**, 96–166 (1945), p. 141; these Rev. **7**, 231]. The author derives a result which is equivalent to the fact that (*) can be satisfied by

$$F(z)=\exp\left[\int^z\{-h(z)\pm m[h'(z)-h^2(z)]^{\frac{1}{2}}\}dz\right],$$

$$f(z)=\exp\left[\int^z\{-h(z)\mp m^{-1}[h'(z)-h^2(z)]^{\frac{1}{2}}\}dz\right],$$

where $h(z)$ is an arbitrary function. *E. Pinney.*

Mitrinovitch, Dragoslav S. Sur un opérateur différentiel. Revue Sci. **89**, 44 (1951).

The author notes that the system

$$\sum_{p=0}^{n}A_p(u, v)u^{(p)}(x)v^{(n-p)}(x)=0,\quad u=F(v),$$

for A_p, F given can be solved by quadratures for $n=2$, and can be transformed to a Riccati equation for $n=3$. *E. Pinney* (Berkeley, Calif.).

Mitrinovitch, D. S. Sur une équation différentielle de Laplace. Bull. Soc. Math. Phys. Macédoine **2**, 109–112 (1951). (French. Serbo-Croatian summary)

The general solution of $xy^{(n)}-mny^{(n-1)}+axy=0$ where m and n are positive integers, $n>2$, is given in a form depending on constants which have to be calculated from a system of linear equations. *W. Feller* (Princeton, N. J.).

Levinson, Norman. A simplified proof of the expansion theorem for singular second order linear differential equations. Duke Math. J. **18**, 57–71 (1951).

The paper is concerned with the Sturm-Liouville equation

$$(1)\qquad \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+(\lambda-q(x))y=0,$$

over the infinite interval $(0, \infty)$. It is assumed that $p(x)>0$, and that $p(x)$, $p'(x)$ and $q(x)$ are real and continuous. If $\varphi(x, \lambda)$ is the solution of (1) for which $\varphi(0, \lambda)=0$, $\varphi'(0, \lambda)=1$, the imposition of a boundary condition linear and homogenous in $y(b)$ and $y'(b)$ at a large positive b, yields a sequence of characteristic values $\lambda_{b,n}$, $n=1, 2, 3, \cdots$, and one of characteristic functions $\varphi(x, \lambda_{b,n})$. The normalizing factor of $\varphi(x, \lambda_{b,n})$ may be designated by $r_{b,n}$. Then if $f(x)\epsilon L^2(0, a)$, and $f(x)\equiv 0$, $x>a$, and $b>a$,

$$\int_0^\infty |f(x)|^2dx=\sum_{n=0}^\infty\left|\int_0^\infty f(x)\varphi(x, \lambda_{b,n})dx\right|^2 r^2_{b,n}.$$

This can be written

$$\int_0^\infty |f(x)|^2dx=\int_{-\infty}^\infty\left|\int_0^\infty f(x)\varphi(x, \mu)dx\right|^2 d\rho_b(\mu)$$

by the use of a step function which increases by $r^2_{b,n}$ at $\mu=\lambda_{b,n}$. It is shown that in this classical relation the passage to the limit $b\to\infty$ can be carried out. *R. E. Langer.*

Langer, Rudolph E. Asymptotic solutions of a differential equation in the theory of microwave propagation. Comm. Pure Appl. Math. **3**, 427–438 (1950).

The asymptotic study, for large values of the constant k, of the differential equation $d^2U/dh^2+k^2[\Lambda+y(h)]U=0$ is of importance in the theory of microwave propagation in an atmosphere whose index of refraction $y(h)$ is a function of the height h. Here Λ is a complex parameter. Pekeris [J. Appl. Phys. **17**, 1108–1124 (1946); these Rev. **10**, 660] derived asymptotic expressions for the solutions of this differential equation. The aim of the present paper is to give an improved and mathematically rigorous asymptotic theory. This is done by applying the author's results [Trans. Amer. Math. Soc. **34**, 447–480 (1932); **67**, 461–490 (1949); these Rev. **11**, 438]. Two cases have to be distinguished according as the roots h_1 and h_0 of the equations $\Lambda+y(h_1)=0$ and $y'(h_0)=0$ are close to each other or not. In each case two sets of asymptotic expressions for a pair of independent solutions are derived, one valid for h near h_1, the other for moderate or large values of $h-h_1$. The asymptotic expressions found previously by Pekeris are not identical with those of the present paper, but the discrepancies tend to zero as $k\to\infty$. *W. Wasow* (Los Angeles, Calif.).

★Schwank, F. Randwertprobleme und andere Anwendungsgebiete der höheren Analysis für Physiker, Mathematiker und Ingenieure. B. G. Teubner Verlagsgesellschaft, Leipzig, 1951. vi+406 pp. $5.67

The topics taken up are indicated by the chapter headings: (I) Boundary value problems, (II) Functions of a complex variable, (III) Partial differential equations, (IV) Integral equations, (V) Calculus of variations, (VI) Difference equations. A reader familiar with only elementary mathematics up to calculus is led very quickly and skillfully to the more advanced mathematical tools needed in applications. New notions are mostly introduced by way of simple special cases and then formulated generally. Theorems are stated correctly, with only one slip noticed by the reviewer

in Cauchy's integral theorem, where simple connectivity is not mentioned. Numerous examples from applied mathematics are discussed, and references to the literature are given for each topic. *F. John* (New Rochelle, N. Y.).

Saltykow, N. Théorie générale des équations aux différentielles totales linéaires par rapport aux variables paramétriques. Acad. Serbe Sci. Publ. Inst. Math. 3, 143–167 (1950).

The first part of this paper contains some remarks about a completely integrable system of equations

$$(1)\quad dx_{m+r}=\sum_{k=1}^{m}\left(\sum_{i=1}^{n-m}A^{k}{}_{ri}x_{m+i}\right)dx_k,\quad r=1,\cdots,n-m,$$

with the $A^k{}_{ri}$ functions of $x_1,\cdots,x_m$. Also the nonhomogeneous case is dealt with. The second part is devoted to the system (1) with the $A^k{}_{ri}$ being constant. The general solution is written as a linear combination of $n-m$ solutions each of the form

$$x_{m+r}=\alpha_r e^{\sum_{k=1}^{m}\theta_k x_k},\quad r=1,\cdots,n-m,$$

θ_k and α_r being constants. Three examples are given.

W. van der Kulk (Providence, R. I.).

Saltikov, N. The general theory of total differential equations linear in the parametric variables. Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka 198, 63–86 (1950). (Serbo-Croatian)

See the paper reviewed above.

Saltikov, N. Problems of integration of total differential equations. Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka 198, 87–104 (1950). (Serbo-Croatian)

In this paper the author takes up again the solution of the system of total differential equations $dx_{m+i}=a^k{}_{ij}x_{m+j}dx^k$ ($k=1,2,\cdots,m$; $i,j=1,2,\cdots,n-m$; summation convention is used) with constant coefficients $a^k{}_{ij}$, that he considered before [C. R. Acad. Sci. Paris **225**, 520–521 (1947); **228**, 1913–1915 (1949); these Rev. **9**, 186; **11**, 111]. He presents three different methods of solution, characterized by the headings: variation of constants, reduction to ordinary differential equations, constant integration factors. From the sketchy presentation the reviewer could not decide whether these methods lead to the general solution in all cases. The paper also contains a theory of integration factors for the general system $dx_{m+i}=X_i{}^k dx_k$, where the $X_i{}^k$ are functions of the variables $x_1,x_2,\cdots,x_m,\cdots,x_n$. The functions $\mu^1,\mu^2,\cdots,\mu^{n-m}$ of these variables are integration factors for the system if the expressions $\mu^i(dx_{m+i}-X_i{}^k dx_k)$ are total differentials. They satisfy the system of partial differential equations of Charpit's type

$$\partial\mu^i/\partial x_k+X_j{}^k(\partial\mu^i/\partial x_{m+j})=-\mu^j(\partial X_j{}^k/\partial x_{m+i}).$$

M. Golomb (Lafayette, Ind.).

Saltikov, N. Lie's generalization of the theory of the last multiplier. Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka 198, 1–16 (1950). (Serbo-Croatian)

To obtain the "last multiplier" for the system of total differential equations $dx_k=\sum_{h=1}^{n}X_k{}^h dt_h$ ($k=1,2,\cdots,m$) the author goes back to the original definition of the multiplier by Jacobi as the Jacobian of a complete system of integrals. If $f_1,f_2,\cdots,f_m$ form a complete system of integrals of the corresponding normal system of partial differential equations then the Jacobian $\Delta=\partial(f_1,\cdots,f_m)/\partial(x_1,\cdots,x_m)$ is shown to satisfy the equations $\partial\Delta/\partial t_h+\sum_{k=1}^{m}\partial(X_k{}^h\Delta)/\partial x_k=0$ ($h=1,2,\cdots,n$), and the quotient Δ/D, where

$$D=\partial(f_1,\cdots,f_{m-1})/\partial(x_1,\cdots,x_{m-1})\neq 0,$$

is an integrating factor for the mth equation of the above system. *M. Golomb* (Lafayette, Ind.).

Saltikov, N. A supplement to the problem of integration of linear partial equations of the first order with a group of infinitesimal transformations. Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka 198, 53–62 (1950). (Serbo-Croatian)

In this note it is shown that some (up to three) integrals can be determined by quadratures for linear partial differential equations in five, six, or seven independent variables with nonintegrable Lie groups of maximal order.

M. Golomb (Lafayette, Ind.).

Saltikov, N. The differential invariants of functional groups of integrals. Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka 198, 17–35 (1950). (Serbo-Croatian)

Let $f_1(x,p), f_2(x,p),\cdots,f_{2\rho}(x,p)$ ($\rho<n$) be a functional group of integrals without distinguished elements of the characteristic system S for the partial differential equation $F(x,p)=0$, where $x=(x_1,x_2,\cdots,x_n)$, $p=(p_1,p_2,\cdots,p_n)$, $p_\nu=\partial z/\partial x_\nu$. The author's method of using these integrals for the calculation of a complete integral of $F=0$ is essentially another variation of Korkin's method. One determines first a canonical group of differential invariants

$$\varphi_1(x,p),\cdots,\varphi_{n-\rho}(x,p),\quad \psi_1(x,p),\cdots,\psi_{n-\rho}(x,p)$$

satisfying the relations $(f_k,\varphi_i)=(f_k,\psi_i)=(\varphi_i,\varphi_j)=(\psi_i,\psi_j)=0$, $(\varphi_i,\psi_j)=\delta_{ij}$ ($k=1,2,\cdots,2\rho$; $i,j=1,2,\cdots,n-\rho$). It follows that there exists a function $\Omega(x,p)$ that satisfies the equations $[\varphi_i,z-\Omega]=0$, $[\psi_i,z-\Omega]=\psi_i$. The equations $x_i'=\psi_i$, $p_i'=\varphi_i$, $z'=z-\Omega$ define a reduced contact transformation by means of which the equation $F(x,p)=0$ is reduced to one in the $n-\rho$ independent variables $\psi_1,\psi_2,\cdots,\psi_{n-\rho}$. To carry out the inverse transformation one still needs to extract ρ integrals of S in involution from the given 2ρ integrals. The method is also applicable to systems of equations $F_\mu(x,p)=0$ ($\mu=1,2,\cdots,m$). *M. Golomb.*

Saltikov, N. Characteristic functions of partial equations of the second order. Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka 198, 37–52 (1950). (Serbo-Croatian)

The two equations of second order

$$F\equiv r+f(x,y,z,p,q,s)=0,\quad G\equiv t+g(x,y,z,p,q,s)=0$$

are in involution in the sense of Darboux-Lie if the four derived equations $D_xF=D_yF=D_xG=D_yG=0$ do not determine the four third order derivatives r_x, $s_x=t_y$, $s_y=t_x$, t_y. For such an involutory system (*) a system of six partial differential equations of first order of Charpit's type for the unknown functions z,p,q,r,s,t and a corresponding characteristic system of ordinary differential equations can be established. Let $y=Y(x,C_1,C_2,\cdots,C_5)$, $z=Z(x,C_1,\cdots,C_5)$, $\cdots$, $t=T(x,C_1,\cdots,C_5)$ be a general integral of the characteristic system. Assuming that $\partial Y/\partial C_5\neq 0$ and that the equation $y=Y$ can be solved by $C_5=\psi(x,y,C_1,\cdots,C_4)$, the function $z=Z(x,C_1,\cdots,C_4,\psi)$ is shown to be a complete integral of (*) if and only if $U_k\neq 0$ ($k=1,2,3,4$) and $U_5=V_5=0$. Here U_k, V_k are the "characteristic functions" defined by

$$U_k=\partial Z/\partial C_k-Q(\partial Y/\partial C_k),\quad V_k=\partial Q/\partial C_k-T(\partial Y/\partial C_k).$$

The author also shows that the conditions for the general

equation of second order $G(x, y, z, \cdots, t)=0$ to be in involution with the general equation of second order $F(x, y, z, \cdots, t)=0$ can be expressed as a pair of first order equations linear in the derivatives of the function G.

M. Golomb (Lafayette, Ind.).

Saltikov, N. Methods of integration of partial equations of the second order with one unknown function. Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka **198**, 105–127 (1950). (Serbo-Croatian)

Besides a historical survey of methods for solving partial differential equations of second order this paper contains an elaboration on a method of solution discussed in earlier publications of the author [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **18**, 810–819 (1932); C. R. Acad. Sci. Paris **195**, 525–527 (1932)]. To the given equation of the form $F\equiv r+f(x, y, z, p, q, s, t)=0$ another equation of the form $G\equiv t+g(x, y, z, p, q, s)=0$ is to be adjoined which is in involution with the first equation either in the sense of Darboux-Lie [see the preceding review] or in the sense of "complete integrability" which means that the functions r, s, t determined by the equations $D_xF=D_yF=D_xG=D_yG=0$ satisfy the conditions $r_y=s_x$, $s_y=t_x$. The former condition restricts the class of equations F, the latter leads to a complicated second order equation for the function g of six independent variables, which can be solved only in very special cases. Once the function g is determined the integral z is found by solving a system of six partial differential equations of first order of Charpit's type. Depending on the generality of the function g a "complete" or "general" integral of $F=0$ may be found.

M. Golomb.

Gillis, Paul P. Équations de Monge-Ampère à six variables indépendantes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **37**, 229–240 (1951).

Let $a_1=z_{11}+z_{22}$, $a_2=z_{33}+z_{44}$, $a_3=z_{55}+z_{66}$, $b_1=z_{13}+z_{24}$, $b_2=z_{15}+z_{26}$, $b_3=z_{35}+z_{46}$, $c_1=z_{14}-z_{23}$, $c_2=z_{25}-z_{16}$, $c_3=z_{36}-z_{45}$, where $z_{ij}=\partial^2 z/\partial x_i\partial x_j$. If in the region D each solution of the differential equation

$$(*)\quad a_1a_2a_3-a_1(b_3^2+c_3^2)-a_2(b_2^2+c_2^2)-a_3(b_1^2+c_1^2) +2b_1b_2b_3-2b_1c_2c_3-2b_2c_1c_3-2b_3c_1c_2=f(x_1, \cdots, x_6)$$

makes $a_2a_3-(b_3^2+c_3^2)>0$ and $a_1f>0$, the differential equation (*) is termed elliptic in D. The author shows that the problem of Dirichlet for an elliptic equation of type (*) cannot have more than one solution if $f(x_1, \cdots, x_6)$ is continuous in D.

F. G. Dressel (Durham, N. C.).

Višik, M. I. On the general form of linear boundary problems for an elliptic differential equation. Doklady Akad. Nauk SSSR (N.S.) **77**, 373–375 (1951). (Russian)

Višik, M. I. On some boundary problems for elliptic differential equations. Doklady Akad. Nauk SSSR (N.S.) **77**, 553–555 (1951). (Russian)

Consider the Hilbert space $\mathfrak{L}_2(D)$ of all square integrable functions in an open bounded domain D of real n-space with a smooth boundary Γ. Consider a second order differential operator $L=[\sum(\partial/\partial x_i)(a_{ik}(x)\partial/\partial x_k)]+[\sum b_i(x)\partial/\partial x_i]+c(x)$, $c(x)\leqq 0$, with sufficiently differentiable coefficients and put $L_0'f=Lf$ whenever $f, Lf\epsilon\mathfrak{L}_2(D)$ and f vanishes in a boundary strip of D. Then L_0' is a densely defined linear operator. Let L_0 be its closure. Similarly for M_0' and M_0, where M is the formal adjoint of L. Let $L_1=M_0^*$ and $M_1=L_0^*$ be the adjoints of M_0 and L_0, respectively. It is clear that $L_0\subset L_1$ and $M_0\subset M_1$. Now consider a linear transformation $\tilde{L}$ such that $L_0\subset\tilde{L}\subset L_1$ which satisfies one of the following conditions: (a) $\tilde{L}^{-1}$ is completely continuous; (b) the three theorems of Fredholm are satisfied for the equations $\tilde{L}f=h$ and $\tilde{L}^*g=h'$; (c) a necessary and sufficient condition that $\tilde{L}f=h$ has a solution is that h be orthogonal to all solutions of $\tilde{L}^*g=0$. Then in each case the domain of definition $\Omega_{\tilde{L}}$ of $\tilde{L}$ can be characterized as the set of all $f\epsilon\Omega_{L_1}$ which satisfy certain boundary conditions involving the values $f|_\Gamma$ and $(\partial f/\partial\nu)|_\Gamma$ of f and its normal derivative at Γ. Let the boundary conditions be given by $(\partial f/\partial\nu)|_\Gamma=Qf|_\Gamma$ where Q is a closed and densely defined operator on $\mathfrak{L}_2(\Gamma)$, and put $Pf|_\Gamma=(\partial u/\partial\nu)|_\Gamma$ where $Lu=0$ in D and $(u-f)|_\Gamma=0$. Then if $Q-P$ satisfies (a), (b), or (c) in $\mathfrak{L}_2(\Gamma)$, the corresponding operator $\tilde{L}$ exists and satisfies (a), (b), or (c), respectively in $\mathfrak{L}_2(D)$. In the second note it is stated that if, in particular, Q is the sum of an elliptic differential operator (on $\mathfrak{L}_2(\Gamma)$) and a bounded operator, then one has the case (a). No proofs are given.

L. Gårding (Lund).

Haack, Wolfgang, und Hellwig, Günter. Die Überführung des Randwertproblems für Systeme elliptischer Differentialgleichungen auf Fredholmsche Integralgleichungen. I. Math. Nachr. **4**, 408–418 (1951).

With a general elliptic system of two linear first order partial differential equations for two unknown functions $u(x^1, x^2)$, $v(x^1, x^2)$ there is associated a Riemannian metric determined by the coefficients of the first derivatives in the differential equations. Given two vector fields with covariant components (α_1, α_2) and $(\bar{\alpha}_1, \bar{\alpha}_2)$, which are orthonormal with respect to the Riemannian metric, one can find suitable linear combinations U, V of u, v, for which the differential equations take the "normal" form

$$U_\alpha-V_{\bar{\alpha}}=AU+BV+C,\quad U_{\bar{\alpha}}+V_\alpha=\bar{A}U+\bar{B}V+\bar{C}.$$

Here for a function S the quantities S_α, $S_{\bar{\alpha}}$ are not ordinary partial derivatives with respect to scalars, but "invariant" derivatives with respect to the two vector fields and are defined by $\partial S/\partial x^i=\alpha_iS_\alpha+\bar{\alpha}_iS_{\bar{\alpha}}$ $(i=1, 2)$. The original system can be brought into normal form by elementary operations, not involving the solution of differential equations. It is proved that U, V do not depend on the choice of the orthonormal vector fields. For a normal system of equations integral identities can be derived, which can be used to obtain a formal reduction of boundary value problems to the solution of a pair of Fredholm integral equations. The form of the Fredholm equation depends on the auxiliary functions used, which may be of the parametrix or of the Green's function type. The question of the existence of the solution and of the Green's function is to be taken up in future communications.

F. John (New Rochelle, N. Y.).

Mihlin, S. G. On equations of elliptic type. Doklady Akad. Nauk SSSR (N.S.) **77**, 377–380 (1951). (Russian)

Let Φ be a selfadjoint elliptic differential operator of second order in n variables. The author derives a Green formula for the equation $-\Phi u+bu=f$, where b and f are given scalars. This formula can be considered as an integral equation for u and it is shown that it is equivalent to the differential equation. Properties such as infinite differentiability and square-integrability are derived.

W. Feller.

Mihlin, S. G. On the algorithm of Schwarz. Doklady Akad. Nauk SSSR (N.S.) **77**, 569–571 (1951). (Russian)

The alternating algorithm for the Dirichlet problem due to H. A. Schwarz is generalized to the equation mentioned in the preceding review.

W. Feller (Princeton, N. J.).

Biegelmeier, Gottfried. Ein Beitrag zur klassischen Diffusionstheorie. Acta Physica Austriaca 4, 278–289 (1950).

The author considers the classical diffusion problem as an initial condition problem in which he seeks the concentration distribution $c=c(x, t)$ which satisfies the differential equation $\partial c/\partial t=D\partial^2 c/\partial x^2$, where D is the diffusion coefficient. The initial condition is $\phi(x)=c(x, 0)=0$ for $|x|\leqq l$, c_0 for $|x|>l$. By the method of separation of variables in which $c(x, t)=f_1(t)f_2(x)$ it is shown that the solution may be expressed in terms of the Gaussian probability integral: that is,

$$c(x, t)=c_0[1+\tfrac{1}{2}\{\psi((x-l)/2(Dt)^{\frac{1}{2}})-\psi((x+l)/2(Dt)^{\frac{1}{2}})\}]$$

where $\psi(x)=2\pi^{-\frac{1}{2}}\int_0^x \exp(-y^2)dy$. As an example, he sets $c_0=D=1$, $|l|=\frac{1}{2}$ and $\phi(x)=0$ for $|x|\leqq\frac{1}{2}$, 1 for $|x|>\frac{1}{2}$, and gives a graphical representation of $c(x, t)$ for particular values of t. This approach is also applied to the diffusion problem in 2 and in 3 dimensions to obtain solutions in terms of the appropriate probability integrals.

C. G. Maple (Washington, D. C.).

Ciliberto, Carlo. Sul problema di Holmgren-Levi per l'equazione del calore. Giorn. Mat. Battaglini (4) 4(80), 1–13 (1951).

L'auteur résoud de manière nouvelle le problème aux limites de l'équation $\partial^2 u/\partial y^2-\partial u/\partial x=0$ pour un domaine limité par deux segments horizontaux et deux arcs $x(y)$ assez réguliers. L'ensemble des données-frontière finies continues f pour lesquelles le problème est résoluble est fermé (pour la topologie de la convergence uniforme) et l'auteur montre que si $\int fd\mu=0$ pour tous ces f, la mesure de Radon μ sur la frontière est $\equiv 0$. Cela entraine d'après le théorème de Hahn-Banach que toute donnée-frontière finie continue est un f.

M. Brelot (Grenoble).

Mitrinovitch, D. S. Sur la solution de Ribaud de l'équation de Fourier. Bull. Soc. Math. Phys. Macédoine 2, 105–107 (1951). (French. Serbo-Croatian summary)

This is a bibliographical note on functions of the type $t^m f(x/t^{\frac{1}{2}})$ which satisfy the simple heat equation. The author points out that most of the results on the nature and application of those functions announced recently by Ribaud [C. R. Acad. Sci. Paris 226, 140–142, 204–206, 444–451 (1948); these Rev. 9, 335], Nordon [ibid. 228, 167–168 (1949); these Rev. 10, 378] and Aubert [ibid. 228, 816–817 (1949); these Rev. 10, 454] were published earlier by Appell [J. Math. Pures Appl. (4) 8, 187–216 (1892)], Goursat [Cours d'analyse mathématique, vol. 3, 5th ed., Gauthier-Villars, Paris, 1942], Horn [Partielle Differentialgleichungen, Berlin, 1929], Kamke [Differentialgleichungen . . . , v. 1, 3d ed., Leipzig, 1944; these Rev. 9, 33] and Buhl [Nouveaux éléments d'analyse, vol. 4, Gauthier-Villars, Paris, 1943].

R. V. Churchill (Ann Arbor, Mich.).

Cadorin, Dante. Sulla formula di Weber relativa all'equazione delle onde sferiche smorzate e forzate. Ist. Veneto Sci. Lett. Arti. Cl. Sci. Mat. Nat. 108, 223–230 (1950).

Derivation of Weber's formula for the solution U of the initial value problem for the equation $\partial^2 U/\partial t^2-\Delta U+kU=X$ from a formula of Tonolo [Rend. Sem. Mat. Univ. Padova 4, 52–66 (1933)], which expresses U in a bounded region S in terms of initial values and of Cauchy data on S.

F. John (New Rochelle, N. Y.).

Smolickiĭ, H. L. Some integral estimates of the derivatives of solutions of the wave equation. Doklady Akad. Nauk SSSR (N.S.) 73, 279–282 (1950). (Russian)

Let Ω be a domain in the $(x)=(x_1, \cdots, x_n)$-space bounded by a sufficiently smooth surface S. Let $u(x, t)$ be l times $(l\geqq 2)$ continuously differentiable in $\bar{\Omega}=\Omega+S$, $t\geqq 0$. Set $\Box u=\Delta u-u_{tt}=f$, $u|_S=\psi$, $u(x, 0)=u_0$, $u_t(x, 0)=u_1$. Let $F_k(t)$, U_0^k, U_1^k be the integrals (over S) of the sums of squares of all partial derivatives up to the order k of f, u_0, u_1, respectively. In each point of S let $(\xi_1, \cdots, \xi_n)$ be a Cartesian coordinate system such that the positive ξ_n-axis is the outer normal to S. Set $\psi_k(t)=\int_S\sum|\partial^m\psi/\partial t^s\partial\xi_{i_1}\cdots\partial\xi_{i_{m-s}}|^2dS$ where the summation is extended over all positive s, i_j with $s+i_1+\cdots+i_{m-s}\leqq k$. Finally set

$$I_{r,\alpha}=\int_\Omega\sum|\partial^{r+\alpha}u/\partial t^\alpha\partial x_{i_1}\cdots\partial x_{i_r}|^2d\Omega$$

where $i_1, \cdots, i_r$ run independently from 1 to n. Let $t_0>0$. The author proves that there exists a function $A_k(t_0)$ and a constant A_k independent of u such that for $0\leqq r+\alpha\leqq k<l$

$$\max_{0\leqq t\leqq t_0} I_{r,\alpha}\leqq A_k(t_0)M^2,$$

where

$$M^2=\max_{0\leqq t\leqq t_0}[U_0^k, U_1^k, \max F_k(t), \psi_{k+1}(t)],$$

and $\sup_{0\leqq t<\infty} I_{r,\alpha}\leqq A_k \max(U_0^k, U_1^k)$ if $f\equiv 0$, $\psi\equiv 0$. These inequalities are used in the note reviewed below.

L. Bers (Los Angeles, Calif.).

Smolickiĭ, H. L. The boundary value problem for the wave equation. Doklady Akad. Nauk SSSR (N.S.) 73, 463–466 (1950). (Russian)

Let Ω be a domain in the $(x)=(x_1, \cdots, x_n)$-space bounded by a smooth surface S. The following problem is considered: $u=u(x, t)$, $x\epsilon\Omega$, $t\geqq 0$, is to satisfy the wave equation (1) $\Delta u-u_{tt}=f(x, t)$, the initial conditions $u(x, 0)=u_0(x)$, $u_t(x, 0)=u_1(x)$ and the boundary condition $u(x, t)|_S=\psi(x, t)|_S$. The given functions (2) f, u_0, u_1, ψ are to satisfy the compatibility conditions for $x\epsilon S$ and $t=0$

$$(3)\qquad \psi=u_0,\quad \psi_t=u_1,\quad \psi_{tt}=\Delta u_0-f,\ \cdots.$$

The surface S is assumed to be such that the equation possesses arbitrarily smooth solutions for arbitrarily smooth f, u_0, u_1, ψ. A sufficient condition is that the maxima of the mth derivatives of eigenfunctions $u_k(x)$ satisfying $\Delta u+\lambda_k u=0$, $u=0$ on S should be $O(\lambda_k^{n+m})$, and this condition is satisfied for infinitely differentiable Liapounoff surfaces. Three existence and uniqueness theorems are stated (and the proofs sketched) for the problem considered. The first assumes the infinite differentiability of the functions (2), and compatibility conditions (3) of all orders. The solution is then infinitely differentiable. In theorem 2 the solution is a "generalized" one. The functions (2) are assumed to have square-integrable mean derivatives of orders α, β, $\beta-1$, γ, respectively, with $1\leqq\delta\leqq\min(\alpha, \beta-1, \gamma)$, and the compatibility conditions are assumed to hold almost everywhere up to the order δ. Theorem 3 deals with "functional solutions". These are defined essentially like L. Schwartz' distributions, except that the comparison functions are finitely (and not infinitely) many times differentiable.

L. Bers.

Gårding, Lars. Linear hyperbolic partial differential equations with constant coefficients. Acta Math. 85, 1–62 (1951).

The equations considered are of the form

$$(1)\qquad q(\partial/\partial x_1, \cdots, \partial/\partial x_n)f(x_1, \cdots, x_n)=0,$$

where $q(\zeta_1, \cdots, \zeta_n) = q(\zeta)$ is a polynomial with complex constant coefficients. The paper is concerned with conditions on the polynomial q that ensure that a solution f of (1) depends continuously on data on a hyperplane. The author succeeds in giving such conditions for the most general equation (1), using a suitable notion of "convergence" of solutions f.

The functions f considered shall belong to the class $C(\infty)$ of functions, which are infinitely differentiable in the whole real Euclidean x-space E_n. A sequence of functions f_k is said to converge strongly to 0 on E_n (or on the hyperplane $x\cdot\xi=0$), if f_k and each derivative of f_k converges uniformly to 0 on every compact subset of E_n (or of $x\cdot\xi=0$). The function q is called hyperbolic with respect to a real vector ξ, if there exists a real number t_0 such that $q(t_0\xi+i\eta)\neq 0$ for all real $t>t_0$ and all real vectors η. The main result of the paper then takes the following form: Hyperbolicity of q with respect to ξ is necessary and sufficient in order that strong convergence of a sequence of solutions f on $x\cdot\xi=0$ implies strong convergence to 0 of the f_k in the whole space E_n.

The hyperbolic equations considered by Herglotz and Petrowsky are also hyperbolic in the sense used here. Hyperbolic character of q with respect to ξ is equivalent with hyperbolic character of the equation given by the terms of highest order, except in certain degenerate cases. In the proof of the main result a number of algebraic difficulties have to be overcome, as is to be expected in a theorem applicable to the most general equation (1). On the analytic side use is made of functional transformations involving the method of fractional integration developed by M. Riesz [Acta Math. **81**, 1–223 (1949); these Rev. **10**, 713]. These transformations also lead to a representation of the solution of the Cauchy problem with Cauchy data prescribed on a hyperplane. The solution of the Cauchy problem with data on a general initial surface is given by the author for special equations, for which q is homogeneous. *F. John.*

Integral Equations

Harazov, D. F. On linear integral equations with generalized kernels of Marty's type. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **17**, 47–60 (1949). (Russian. Georgian summary)

Let x, y be points in the bounded closed region T of n-dimensional Euclidean space. The author calls the function $G(x, y; \lambda) = \sum_{k=0}^{m} \lambda^k G_k(x, y)$ a generalized kernel of Marty's type if a) $G_k(x, y)$ $(k=0, 1, \cdots, m)$ is a complex-valued function in $L^2(T)$, b) $G_k(x, y)$ is of Marty's type, more precisely, there exists a positive-definite Hermitian kernel $H(x, y)\epsilon L^2(T)$ (H independent of k) such that $\int_T H(x, t)G_k(t, y)dt$ is Hermitian. Defining characteristic values for the equation $u(x) - \int_T G(x, y; \lambda)u(y)dy = 0$ in the obvious way, sufficient conditions are derived for the existence of at least one real characteristic value and for the nonexistence of nonreal characteristic values in the angular sectors $|\arg \lambda| < \pi/(m-1)$, $|\arg \lambda - \pi| < \pi/(m-1)$. For the case $m=2$ the following theorem results: If $G_0(x, y)$ has no positive characteristic values $\leqq 1$, and $G_2(x, y)$ has only positive characteristic values then the equation has at least one characteristic value and all characteristic values are real. *M. Golomb* (Lafayette, Ind.).

Šerman, D. I. On a case of regularization of singular equations. Akad. Nauk SSSR. Prikl. Mat. Meh. **15**, 75–82 (1951). (Russian)

In an earlier note [Doklady Akad. Nauk SSSR (N.S.) **59**, 647–650 (1948); these Rev. **9**, 442] the author indicated a new method of transforming the singular equation (in the sense of principal values)

$$(1)\quad A(t_0)w(t)+\frac{1}{\pi i}B(t_0)\int_L \frac{w(t)}{t-t_0}dt+\int_L w(t)G(t, t_0)dt=f(t_0)$$

into a Fredholm integral equation. Here L is a closed, suitably smooth curve, bounding a bounded, simply connected domain; t, t_0 are points on L; A, B, G, f are given on L and are of Hölder classes. This transformation was possible when the functions $A\pm B$ vanish at some points of L. In this earlier work it was assumed that $A(t_0)$, $B(t_0)$, $G(t, t_0)$, $f(t_0)$ are analytic in t_0 at those points. In the present work the author studies (1), adapting the method of his preceding note to the regularisation of (1) under less restrictive conditions on the coefficients involved. *W. J. Trjitzinsky.*

Vekua, N. P. Hilbert's boundary problem and systems of singular integral equations in the case of piecewise smooth contours. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **17**, 29–40 (1949). (Georgian. Russian summary)

In this note we generalize all the results obtained in an earlier work [Mushelišvili and Vekua, Trudy Tbiliss. Mat. Inst. **12**, 1–46 (1943); these Rev. **6**, 272] for the case of smooth contours to the case of piecewise smooth contours. *Author's summary.*

Hvedelidze, B. V. On linear singular integral equations with a singular kernel of Cauchy type. Doklady Akad. Nauk SSSR (N.S.) **76**, 367–370 (1951). (Russian)

The author makes use of the notation and definitions of an earlier note [same Doklady N.S. **76**, 177–180 (1951); these Rev. **12**, 817]. On the basis of this note theorems are established, analogous to those of Noether, for linear integral equations with the Cauchy kernel and with integrations in the sense of principal values. *W. J. Trjitzinsky.*

Scott, W. T. On a difference equation method in cosmic-ray shower theory. Physical Rev. (2) **80**, 611–615 (1950).

The theory of the cascade theory of cosmic-ray showers leads to a pair of integro-differential equations for the functions $P(x, t)$ and $\gamma(x, t)$ which give, respectively, the mean energy spectrum of electrons and photons at depth t. By applying to these equations a Laplace transformation in t and a Mellin transformation in x in the manner

$$g(p, s)=\int_0^\infty dt\, e^{-st}\int_0^\infty dx\, x^p P(x, t)$$

and

$$\theta(p, s)=\int_0^\infty dt\, e^{-st}\int_0^\infty dx\, x^p \gamma(x, t),$$

the author derives the pair of difference equations:

$$(1)\quad pg(p-1, s)+G(p, s)g(p, s)=\varphi(p),$$

$$(2)\quad (s+D)\theta(p, s)=C(p)g(p, s)+\varphi_\gamma(p),$$

where $G(p, s)$ and $C(p)$ are known functions, D is a constant and $\varphi(p)$ and $\varphi_\gamma(p)$ are the Mellin transforms of the initial spectrum of electrons and photons. Equations (1) and (2) are to be solved in such a manner that they admit proper

inverse Mellin transforms. The solution for $g(p, s)$ is found in the form

$$g(p, s)=\sum_{m=0}^{\infty}(-1)^m\varphi(p+m+1)\frac{\Gamma(p+1)}{\Gamma(p+m+2)}Q(p+1, m, s)$$

where

$$Q(p+1, m, s)=G(p+1, s)G(p+2, s)\cdots G(p+m, s).$$

The function $Q(p+1, m, s)$ satisfies the recurrence relations $G(p, s)Q(p+1, m, s)=Q(p, m+1, s)$ and

$$G(p+m, s)Q(p, m, s)=Q(p, m+1, s)$$

and is given by

$$Q(p+1, m, s) =G(b, s)^m\prod_{j=0}^{\infty}\frac{G(p+j+1, s)}{G(p+j+m+1, s)}\left\{\frac{G(b+j+1, s)}{G(b+j, s)}\right\}^m.$$

By taking the inverse Mellin and Laplace transforms we find

$$P(x, t)=\frac{-1}{(2\pi i)^3}\int_{C_s}ds\, e^{st}\int_{C_p}dp\, x^{-p-1} \times\int_{C_\sigma}d\sigma\frac{\pi}{\sin\pi\sigma}\frac{\Gamma(p+1)}{\Gamma(p+\sigma+2)}Q(p+1, \sigma, s)\varphi(p+\sigma+1)$$

where C_σ is a contour parallel to the imaginary axis cutting the real axis between 0 and -1 and C_s and C_p are contours parallel to the imaginary axis in the s- and the p-planes, respectively, and each taken to the right of all the singularities. The solution for $\gamma(x, t)$ is similarly expressed. The author shows how from the foregoing solution the earlier solutions of Snyder [Physical Rev. (2) 76, 1563–1571 (1949)] and Bhabha and Chakrabarty [ibid. 74, 1352–1363 (1945)] for the particular case $P(x, 0)=\delta(x-x_0)$ and $\gamma(x, 0)=0$ can be derived. *S. Chandrasekhar.*

Bernstein, I. B. Improved calculations on cascade shower theory. Physical Rev. (2) 80, 995–1005 (1950).

In the solution of the diffusion equations of the shower theory given as an inverse Laplace-Mellin transform [see the preceding review] the function $R(E, E')$ giving the elementary probabilities per unit length of pair production and bremsstrahlung processes (E is the initial energy and E' the final energy) is generally approximated by a function $R_0(E, E')$ which is homogeneous in E/E': this is the so-called "completely screened approximation". In a higher approximation $R(E, E')$ can be represented by an interpolation formula of the type

$$R(E, E')=R_0(E, E')/[1+K|E'/E(E-E')|]$$

where K is a constant. In this paper the correction to the solution of the diffusion equations resulting from the use of this more accurate formula for the basic probability function is considered. Representing the solution derived on $R_0(E, E')$ for incident monoenergetic electrons (E_0) by $P_0(E, t, E_0)$, the author develops an iteration scheme for improving the solution. Letting $N(E, t, E_0)=\int_E^\infty dE\, P(E, t, E_0)$, the author finds that the first order correction to the average number of electrons $N(0, t, E_0)$ at thickness t, given by the completely screened approximation is represented by the inverse Laplace transform of a known function. The latter inversion is performed numerically and tables are provided in terms of which the corrections can be readily calculated for all light elements when the incident particle is either a single photon or a single electron. *S. Chandrasekhar.*

Janssens, P., et Gribaumont, A. Sur la répartition des neutrons dans un modérateur. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 36, 616–626 (1950).

The problem of the distribution of fast neutrons in space and of energy in an infinite moderator with a point source at the center and in the absence of captures is considered by a generalization of a method due to P. Langevin [Ann. Physique (11) 17, 303–317 (1942); these Rev. 5, 55]. The problem is also solved by considering the generalization of an integral equation due to S. Flügge [Z. Physik 111, 109–124 (1938)]. It is shown that both methods give the same results. The method of Flügge is further generalized to include capture. *S. Chandrasekhar.*

Holte, Gunnar. On the space energy distribution of slowed-down neutrons. Ark. Fys. 2, 523–549 (1951).

In the theory of the space-energy distribution of slowed down neutrons in a moderator (containing atomic nuclei of mass M) the expression for the number of neutrons $F_0(R, E)$ of energy E at a distance R (measured in units of the mean free path) when Q neutrons per unit time are emitted at the center with an energy E_0, is expressed as the Fourier-Laplace transform of a function $f_0(\omega, \eta)$ in the form

$$(*)\quad F_0(R, E)=-\frac{1}{2\pi^2R}\int_{\eta_1-i\infty}^{\eta_1+i\infty}d\eta\, e^{x\eta}\int_{-\infty}^{+\infty}d\omega\, \omega e^{i\omega R}f_0(\omega, \eta),$$

where $x=\log E_0/E$ and the path of integration in the η-plane is a straight line to the right of the singularities of the integrand. Waller [Ark. Mat. Astr. Fys. 34A, no. 3 (1947); these Rev. 8, 587] has given an expression for $f_0(\omega, \eta)$ as a continued fraction in ω and a sequence of functions $h_n(\eta)$. From Waller's expression it follows that the principal contribution to the integral over ω comes from a pole ω_0 between 0 and $+i$ on the imaginary axis and that

$$\text{Residue}\{f_0\,(\omega, \eta)\}_{\omega=\omega_0(\eta)}=\frac{Q}{4\pi E_0}\frac{3h_1}{2\omega_0(\eta)}A(\eta),$$

where A as a function of $\theta=\eta-1$ has a series expansion of the form $(1+\sum\alpha_r\theta^r)$ for $\theta\to 0$. Accordingly,

$$F_0(R, E)\simeq\frac{Q}{8\pi^2}\frac{3h_1}{ER}\frac{1}{i}\int_{L_\theta}d\theta\,[\exp\{x\theta-R|\omega_0(\theta)|\}]A(\theta),$$

where the path of integration L_θ is a straight line in the right half-plane parallel to the imaginary axis. This last integral is evaluated by the method of steepest descents and the author finally obtains an expression of the form

$$F_0(R, E)=\frac{Q}{Ex^{\frac{1}{2}}}\mu e^{-\nu R}$$

where ν and to a first approximation also μ are functions of R/x only. Explicit expressions are given for these functions and applications of the theory to a moderator containing deuterium ($M=2$) and carbon ($M=12$) are made.
S. Chandrasekhar (Williams Bay, Wis.).

Functional Analysis, Ergodic Theory

✱Schwartz, Laurent. Théorie des distributions. Tome II. Actualités Sci. Ind., no. 1122 = Publ. Inst. Math. Univ. Strasbourg 10. Hermann & Cie., Paris, 1951. 169 pp.

This volume is a detailed treatment of the harmonic analysis of distributions on Euclidean spaces or toruses, and

like volume I [Théorie des distributions. Tome I. Actualités Sci. Ind., no. 1091 = Publ. Inst. Math. Univ. Strasbourg 9, Hermann et Cie., Paris, 1950; these Rev. 12, 31] is unusually clearly written and well organized. A summary of the major results of the volume appeared in Ann. Univ. Grenoble. Sect. Sci. Math. Phys. (N.S.) 23, 7–24 (1948); these Rev. 10, 36. The results are qualitatively similar on the whole to the well known ones for functions, but serve to extend these results to maximal (though not maximum) domains of applicability. The techniques are partly standard, and partly based on a novel use of the theory of topological linear spaces, which use is appropriate in the present connection in view of the introduction of at least 14 distinct types of spaces of distributions. While the present approach is more systematic and algebraic than the common ones in harmonic analysis on the reals, its analytical force does not seem very substantially greater, and for the most part it would seem to amount to a much greater improvement in form than in essential content. On the other hand, the treatment of partial differential equations with constant coefficients given in the present volume is significantly simpler and more intelligible than the conventional ones.

The first of the two chapters concerns convolutions of distributions, which are first defined for distributions with compact supports by the formula $(S*T)(f)=(S_a\times T_b)f(a+b)$, where $\times$ indicates the direct product, and are shown to have the expected properties. Later the convolution is defined for distributions with supports A and B such that $A\cap(C-B)$ is compact for all compact sets C, an extension required for the treatment of differential equations. An intensive study is made of the regularization of a distribution T by an infinitely differentiable function f, this being defined as the convolution of T with f and being itself an infinitely differentiable function. Convergence of distributions and boundedness of a set of distributions are shown to be equivalent to corresponding properties of the regularizations of the distributions, and by these and related results the investigation of harmonic properties of distributions is reduced in considerable part to the corresponding investigation for functions. The limitations imposed on a distribution by restrictions of various types on its derivatives are studied, and classes of distributions analogous to the classical L_p spaces and to the Stepanoff almost periodic functions are treated. The chapter ends with an interesting treatment of equations such as $A*T=B$, where A and B are given distributions, based on the use of elementary solutions, this being in the case of the equation cited a distribution E such that $A*E$ is the Dirac "function". Applications are made to the analyticity of solutions of homogeneous elliptic equations and to a generalized notion of superharmonic function.

The second chapter treats the Fourier transform for distributions. A periodic distribution always has a Fourier series which converges to the distribution, and any sequence of coefficients which grow sufficiently slowly is the Fourier coefficient sequence of some distribution. An arbitrary distribution on Euclidean space will not have a Fourier transform, but a "temperate" (originally, "spherical") distribution has a Fourier transform, and the class of all such is mapped bicontinuously and univalently onto itself by the Fourier transform. Among other results, the Parseval formula holds, essentially as a matter of definition of the Fourier transform. Fourier transforms of distributions with compact supports and "positive definite" distributions are characterized, in extension of well-known results of Paley and Wiener and of Bochner. The chapter closes with a number of interesting applications to partial differential equations of all types with constant coefficients, and to other equations of the convolution type described in the preceding paragraph.

The choices of the linear topological spaces involved in the foregoing are limited in various ways, but in many respects formally similar results could be obtained with rather different spaces. For example, for a number of purposes in harmonic analysis it would seem possible and even convenient to replace infinite (local) differentiability by infinite differentiability in L_2.

The book has excellent summaries preceding each chapter, very good indexes of nomenclature and notations, and includes a bibliography pertinent to the first as well as to the present volume. *I. E. Segal* (Princeton, N. J.).

***Lévy, Paul. Problèmes concrets d'analyse fonctionnelle. Avec un complément sur les fonctionnelles analytiques par F. Pellegrino.** 2d ed. Gauthier-Villars, Paris, 1951. xiv+484 pp. 4000 Francs.

This volume is a second edition of the author's Leçons d'analyse fonctionelle [Gauthier-Villars, Paris, 1922]. Changes are limited mainly to the first part which concerns the foundations of the functional calculus. The earlier edition appeared when notions of functionals were still in the developmental stage and so it was found necessary to devote space to the introduction of the pertinent ideas, and justification of the forms of operations to be considered. The new edition takes most of these things for granted, for instance assumes a knowledge of Lebesgue integrability, as well as metric space notions. There is still a strong preference for linear functionals expressible in the form of an integral inner product $\int x(t)\alpha(t)dt$ and corresponding forms for multilinear functionals and transformations. The second and third part of this volume are essentially unchanged from the first edition. New is a fourth part, devoted to an exposition of the Fantappiè theory of analytic functionals by F. Pellegrino. This section is expository in character giving a survey of the results obtained by Fantappiè and his pupils in this field. As a consequence many results are mentioned without proof. The chapter headings of this portion are as follows: I. The field of definition of analytic functionals. II. Analytic functionals in general. III. Linear analytical functionals. IV. Mixed analytic functionals. V. Some applications of linear functionals. VI. Nonlinear analytic functionals. There is an extensive bibliography limited to the subject of analytic functionals at the end of this section. This section should prove valuable in giving an introduction to and an extensive survey of this interesting part of functional analysis.

T. H. Hildebrandt (Ann Arbor, Mich.).

Dieudonné, Jean. Sur les espaces de Köthe. J. Analyse Math. 1, 81–115 (1951).

Generalizing Köthe's "vollkommene Räume" [Math. Nachr. 4, 70–80 (1951); these Rev. 12, 615], the author defines a "Köthe space" as follows: Let E be a locally compact space which is the union of denumerably many compact sets, μ a nonnegative Radon measure on E, and Ω the vector space of equivalence classes of functions locally integrable on E (i.e., integrable on every compact set). (Two functions are equivalent if they are almost everywhere equal on every compact set.) Let Ω be given a "strong" topology by a family of semi-norms: $N_g(f)=\int g|f|d\mu$, $g\epsilon L_+(E)$. For $\Gamma\subset\Omega$, the set Λ of functions $f\epsilon\Omega$ for which fg is integrable for every $g\epsilon\Gamma$ is called the Köthe space of the defining set Γ. The set Λ^* whose defining set is Λ is called the associated Köthe

space and in terms of the bilinear form $(f, g) = \int fg d\mu$ they are weakly dually related. The weak topology of Λ^* is the weak functional topology relative to the topology of Λ induced by Ω. An essential tool is the representation of Λ as the intersection of spaces L'_g, where L'_g is the Köthe space whose defining set is a $g \epsilon \Lambda^*$, $g \geqq 0$. The "strong" topology $\beta(\Lambda, \Lambda^*)$ for Λ is the topology of uniform convergence on all weakly bounded sets of Λ^*, and, relative to $\beta(\Lambda, \Lambda^*)$, Λ is complete. The topology $\tau(\Lambda, \Lambda^*)$ of uniform convergence on weakly compact sets of Λ^* is equivalent to that of uniform convergence on convex weakly compact sets and is the strongest relative to which Λ^* is the dual of Λ. With respect to $\tau(\Lambda, \Lambda^*)$ Λ is complete. Then β and τ are equivalent if and only if bounded sets in Λ^* are relatively weakly compact. A limit set G in a locally convex T_2 space F is one for which $\lim_{n\to\infty} \sup_{x\epsilon G} |(x, x_n')| = 0$ for any $x_n' \to 0$, weakly, in the dual F' of F. A set in Λ is relatively compact for τ if and only if it is a limit set. Further extensions and generalizations of Köthe's work conclude the paper. The first section and parts of the second constitute a convenient summary of the theory of weak topologies in linear spaces, measures, etc.

B. R. Gelbaum (Minneapolis, Minn.).

Pettis, B. J. On the continuity of parametric linear operations. Proc. Amer. Math. Soc. 2, 455–457 (1951).

Let H be a subset of a group with a topology, X a Banach space, $B(X)$ the set of all continuous linear operators from X to itself and T_h a function from H to $B(X)$ such that $T_h(T_kx) = T_{h+k}x$ if $h, k, h+k \epsilon H$. The author proves a theorem giving conditions under which weak continuity of T_h from H to $B(X)$ implies strong continuity. Although the conditions are more involved, the theorem is of the same general type as those of Hille [Functional Analysis and Semi-Groups, Amer. Math. Soc. Colloq. Publ., vol. 31, New York, 1948, pp. 183–184; these Rev. 9, 594] but makes use of only topological notions with no measure theoretic concepts involved.

R. E. Fullerton (Madison, Wis.).

Zaanen, A. C. Characterization of a certain class of linear transformations in an arbitrary Banach space. Nederl. Akad. Wetensch. Proc. Ser. A. 54 = Indagationes Math. 13, 87–93 (1951).

Let E be a Banach space, and let E^* be its conjugate space. If $f \epsilon E$, $f^* \epsilon E^*$, write (f, f^*) for $f^*(f)$, i.e. for the value of the functional f^* at f. A bounded transformation A of E into E^* is called "pseudo-selfadjoint" whenever $(f, Ag) = (g, Af)$ for all $f, g \epsilon E$; it is called "pseudo-positive" if, moreover, $(f, Af) \geqq 0$ for all $f \epsilon E$. Let C denote the class of all linear bounded transformations K of E into itself which satisfy the following conditions: (i) K^p is completely continuous for an integer $p \geqq 1$, (ii) there is a bounded pseudo-positive transformation H of E into E^* such that HK is pseudo-selfadjoint and $\neq 0$, (iii) if $Kf = \lambda f \neq 0$, then $(f, Hf) \neq 0$. The following theorem is proved: In order that K belong to class C, it is necessary and sufficient that the condition (i) and the following two conditions be satisfied: (ii') K has at least one characteristic value $\neq 0$, (iii') all characteristic values $\neq 0$ of K are real and are simple poles of the resolvent $(K - \lambda I)^{-1}$.

B. Sz.-Nagy (Szeged).

Peck, J. E. L. An ergodic theorem for a noncommutative semigroup of linear operators. Proc. Amer. Math. Soc. 2, 414–421 (1951).

Let $\mathfrak{S}$ be a semi-group in the totality $\mathfrak{B}$ of bounded linear transformations S acting on a Banach space $\mathfrak{X}$. As suggested by Alaoglu and Birkhoff [Ann. of Math. (2) 41, 293–309 (1940); these Rev. 1, 339], W. Eberlein [Trans. Amer. Math. Soc. 67, 217–240 (1949); these Rev. 12, 112] called $\mathfrak{S}$ ergodic if there is a directed set of operators $\{T_\gamma | \gamma \epsilon \Gamma\}$ in $\mathfrak{B}$ with the properties: (a) $T_\gamma x \epsilon \mathrm{Cl}((\mathrm{Conv}\,\mathfrak{S})x)$ = (strong closure of the convex combinations of Sx, $S \epsilon \mathfrak{S}$) for $\gamma \epsilon \Gamma$, $x \epsilon \mathfrak{X}$; (b) $\sup_\gamma \|T_\gamma\| < \infty$; (c) if $S \epsilon \mathfrak{S}$, then strong $\lim_\gamma (ST_\gamma - S)x = 0$, strong $\lim_\gamma (T_\gamma S - S)x = 0$. An element $x \epsilon \mathfrak{X}$ is ergodic with respect to this ergodic semi-group if there exists a $y \epsilon \mathrm{Cl}((\mathrm{Conv}\,\mathfrak{S})x)$ such that $Sy = y$, $S \epsilon \mathfrak{S}$. The author proves the result: Let a semi-group $\mathfrak{S}$ be totally bounded in the strong topology of operators; then $\mathfrak{S}$ is ergodic and every element $x \epsilon \mathfrak{X}$ is ergodic with respect to $\mathfrak{S}$, when $\mathfrak{S}$ is pseudo-commutative: if S_1, $S_2 \epsilon \mathfrak{S}$, then there exists S_3, S_4, S_5, $S_6 \epsilon \mathfrak{S}$ such that $S_1S_3 = S_2S_4$, $S_1S_2 = S_5S_1 = S_2S_6$. This result may be compared with those due to Alaoglu and G. Birkhoff, Eberlein, and Day [Trans. Amer. Math. Soc. 51, 399–412 (1942); these Rev. 4, 14] who treated the cases of reflexive space $\mathfrak{X}$ and weakly compact $\mathfrak{S}$, with or without some sort of commutativity of $\mathfrak{S}$. The proof of the result is obtained by a fixed point theorem of S. Kakutani [stated without a detailed proof in Proc. Imp. Acad. Tokyo 14, 242–245 (1938)], a complete proof of which is reproduced in the present paper.

K. Yosida (Nagoya).

Kantorovič, L. V. The principle of the majorant and Newton's method. Doklady Akad. Nauk SSSR (N.S.) 76, 17–20 (1951). (Russian)

The author extends his treatment of Newton's method of approximation for equations in normed linear spaces [cf., e.g., Trudy Mat. Inst. Steklov 28, 104–144 (1949); these Rev. 12, 419] to linear spaces whose elements are valued by elements of another partially ordered linear space. A statement of the details would entail setting forth a number of definitions. The idea is to have a transformation $y = P(x)$ between two linear spaces X and Y compared with a transformation $w = Q(z)$ between the spaces Z and W which value X and Y, respectively. Then, when Q behaves properly as a majorant for P, and when Newton's method converges for Q, it also converges for P.

J. V. Wehausen.

Citlanadze, È. S. On extrema of functionals in linear spaces. Doklady Akad. Nauk SSSR (N.S.) 76, 797–800 (1951). (Russian)

The results announced in this paper seem to be essentially the same as those of an earlier paper (abstract of a lecture before the Moscow Mathematical Society) [Uspehi Matem. Nauk (N.S.) 5, no. 4(38), 141–142 (1950); see also the same Doklady (N.S.) 57, 879–881 (1947); 71, 441–444 (1950); these Rev. 12, 110; 9, 447; 11, 670]. Proofs are sketched.

J. V. Wehausen (Providence, R. I.).

Keldyš, M. V. On the characteristic values and characteristic functions of certain classes of non-self-adjoint equations. Doklady Akad. Nauk SSSR (N.S.) 77, 11–14 (1951). (Russian)

The equations considered are of the form $y = L(\lambda)y$, $L(\lambda) = K_0 + \lambda K_1 + \cdots + \lambda^n K_n$, where λ is a complex number, y an element of the Hilbert space S, and $K_0, K_1, \cdots, K_n$ completely continuous operators. The number $\lambda = c$ is said to be a characteristic value with the characteristic element y_0 and the conjugate elements $y_1, y_2, \cdots, y_k$ if

$$y_0 = L(c)y_0, \cdots, y_k = L(c)y_k + \frac{1}{1!}\frac{\partial L(c)}{\partial c}y_{k-1} + \cdots + \frac{1}{k!}\frac{\partial^k L(c)}{\partial c^k}y_0.$$

From properly chosen characteristic and conjugate elements $y_0, y_1, \cdots, y_k$ one forms the n systems of characteristic and conjugate elements

$$u_{\nu k}=[(d^\nu/dt^\nu)e^{\alpha t}(y_k+y_{k-1}t/1!+\cdots+y_0t^k/k!)]_{t=0}$$

($\nu=0, 1, \cdots, n-1$). Then the system of characteristic and conjugate elements is said to be n-complete if any n-tuple $(f_0, f_1, \cdots, f_{n-1})$ of elements of S can be obtained as the limit of linear combinations $f_\nu^{(N)}=\sum a_{kj}^{(N)}u_{\nu k}^j$ with coefficients independent of ν. One of the stated results is: If H is non-singular, self-adjoint, and some power of H is of finite norm, if A is completely continuous, $B_1, B_2, \cdots, B_{n-1}$ bounded, then the system of characteristic and conjugate elements of $y=(A+\lambda HB_1+\cdots+\lambda^{n-1}HB_{n-1}+\lambda^nH)y$ form an n-complete system. The characteristic values of this equation approach the rays $\arg \lambda=k\pi/n$ asymptotically. More precise asymptotic distributions of the characteristic values are found for more restricted H. The results are applied to boundary value problems for the elliptic partial differential equation

$$\sum p_{ik}\partial^2u/\partial x_i\partial x_k+\sum q_i\partial u/\partial x_i+(r_0+\lambda r_1+\cdots+\lambda^n r_n)u=0$$

and to boundary value problems of ordinary differential equations. *M. Golomb* (Lafayette, Ind.).

Krasnosel'skiĭ, M. A. The continuity of the operator $\mathfrak{f}u(x)=f[x, u(x)]$. Doklady Akad. Nauk SSSR (N.S.) 77, 185–188 (1951). (Russian)

Let $f(x, u)$ be defined for all real u and for $x\epsilon G$, a measurable set, and assume f is continuous for almost all fixed $x\epsilon G$ and is measurable for u fixed. The continuity of the non-linear operator $\mathfrak{f}u(x)=f[x, u(x)]$ under the hypothesis that G is of finite measure was investigated by the author and some others before [same Doklady (N.S.) 73, 13–15 (1950); these Rev. 12, 111]. In this paper it is assumed that G is of infinite measure and it is proved that if $\mathfrak{f}$ transforms every function of L^{p_1} into a function of L^{p_2} then $\mathfrak{f}$ is continuous (in the obvious sense) and bounded on every sphere. The author observes that a theorem of Vaĭnberg [ibid. 73, 253–255 (1950); these Rev. 12, 111] to the effect that $\mathfrak{f}$ is continuous if compact is pointless since $\mathfrak{f}$ can be compact only if constant. *M. Golomb* (Lafayette, Ind.).

✱Hamburger, H. L., and Grimshaw, M. E. Linear Transformations in n-Dimensional Vector Space. An Introduction to the Theory of Hilbert Space. Cambridge, at the University Press, 1951. x+196 pp.

As the title and subtitle of this tract indicate, the material it covers is well known and can be found in many books. The authors manage, nevertheless, to present several facts, developments, and points of view that it would be difficult for a student to find in, or deduce from, the literature easily available to him. The novelty is achieved by a deliberate change of emphasis, opposed to some of the modern tendencies.

As is proper in a treatment whose motivation is the infinite-dimensional case, none of the proofs relies on the theory of determinants. The existence of proper values for Hermitian transformations is proved by a limiting argument, and analytic methods are freely employed throughout the book. The resolvent gets its share of attention, and a few sections are devoted to applications with a physical flavor. The longest of the five chapters into which the book is divided contains a detailed discussion of the theory of elementary divisors and the Jordan canonical form. It is natural in this connection to expect to see methods that apply in the finite-dimensional case only, but they are kept to a minimum. The crucial argument is made to depend on the existence, for any linear transformation, of an annihilating polynomial.

The authors have chosen to formulate the basic definitions in terms of coordinates. On page 1, for instance, a vector is defined as an ordered system of n complex numbers. The effects of this decision are visible throughout and are seen most clearly in the fifth and last chapter (Vector spaces with positive Hermitian metric forms). Since the concepts of vector space and unitary space have not been defined before, it becomes necessary at this point to start all over again and to re-define and re-examine such concepts as linear manifold and Hermitian transformation.

The quality of the book is such that every student of the subject, whether he agrees or disagrees with the point of view of the authors, will find it a valuable addition to his library. *P. R. Halmos* (Montevideo).

✱Julia, Gaston. Les racines, carrées et $n^{\text{ièmes}}$, des opérateurs hermitiens dans l'espace unitaire ou hilbertien. Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 195–198. Centre National de la Recherche Scientifique, Paris, 1950.

A survey of the elementary remarks on the subject described in the title that the author has been making in a long series of notes. *P. R. Halmos* (Montevideo).

Putnam, C. R. On commutators of bounded matrices. Amer. J. Math. 73, 127–131 (1951).

If A, B are bounded linear transformations in Hilbert space, then $C=AB-BA$ cannot be a constant multiple, other than zero, of the identical transformation. This has been proved by A. Wintner in the case where A and B are Hermitian [Physical Rev. (2) 71, 738–739 (1947); these Rev. 8, 589], and by H. Wielandt in the general case [Math. Ann. 121, 21 (1949); these Rev. 11, 38]; it is shown in this note, that Wintner's argument may be used also to prove the general statement. Wintner raised the question whether, for any bounded linear A, B, the commutator C satisfies (1) $\inf_{\|x\|=1}|(Cx, x)|=0$, which would be a stronger result than the above. (1) is true for finite matrices, but in the general case it is not yet proved. In the present note, the following particular result is obtained: Let $M(\lambda)$ ($\lambda\neq0$) denote the greatest point of the spectrum of the Hermitian transformation $\lambda(AA^*-A^*A)-\lambda^{-1}(BB^*-B^*B)$, then

$$\inf_{\|x\|=1} |(Cx, x)|\leqq\tfrac{1}{2} \max \{0, \inf_{\lambda\neq0} M(\lambda)\}.$$

It results that (1) is true in particular if AA^*-A^*A and BB^*-B^*B are semidefinite. *B. Sz.-Nagy* (Szeged).

Magnus, Wilhelm. On the spectrum of Hilbert's matrix. Amer. J. Math. 72, 699–704 (1950).

It has been shown by Hilbert that the infinite matrix (α_{nm}) where $\alpha_{nm}=(n+m+1)^{-1}$ ($n, m=0, 1, \cdots$) has the bounds 0 and π, i.e. that (1) $0\leqq\sum_{n,m=0}^\infty\alpha_{nm}x_nx_m\leqq\pi\sum_{n=0}^\infty x_n^2$ for all real x_n. In this paper it is shown that this matrix has the purely continuous spectrum $[0, \pi]$. For the integral equation with kernel $\alpha(s, t)=(s+t)^{-1}$ ($0<s, t<\infty$) the corresponding fact has been established by Carleman [Sur les équations intégrales singulières à noyau réel et symétrique, Uppsala, 1923, p. 169], and the author remarks that the partial statement that the spectrum of (α_{nm}) fills up $[0, \pi]$ should follow from Carleman's theorem, from Hilbert's

original proof of (1) [see H. Weyl, Dissertation, Göttingen, 1908, pp. 83–86] and from a result due to H. Weyl [Rend. Circ. Mat. Palermo 27, 373–392, 402 (1909)]. To prove his theorem, the author reduces it to the following theorem on integral equations: If $0 \leqq \lambda \leqq \pi$, then neither of the integral equations $\int_0^1 \{g(t)/(1-ts)\}dt = \lambda g(s)$, $\int_0^1 \{h(t)/(1-ts)\}dt = \lambda h(s)+1$ has a solution $g(s)=\sum_{n=0}^{\infty} g_n s^n \neq 0$, $h(s)=\sum_{n=0}^{\infty} h_n s^n$ with coefficients g_n, h_n which are real and such that $\sum g_n^2$, $\sum h_n^2$ are convergent. This is shown, in a quite intricate way, by passing first to the iterated integral equations, then using Laplace transforms and Stieltjes' inversion formula.

B. Sz.-Nagy (Szeged).

Štraus, A. V. On the theory of the generalized resolvent of a symmetric operator. Doklady Akad. Nauk SSSR (N.S.) 78, 217–220 (1951). (Russian)

A continuation of an earlier paper [same Doklady (N.S.) 71, 241–244 (1950); these Rev. 11, 600]. The author shows that every generalized resolvent R_λ of a closed Hermitian operator whose domain is dense in a Hilbert space $\mathfrak{H}$, has the form $R_\lambda = (A_{F(\lambda)} - \lambda I)^{-1}$, where $\mathfrak{J}\lambda \cdot \mathfrak{J}\lambda_0 > 0$, $F(\lambda)$ is any regular operator function from $\mathfrak{M}(\bar{\lambda}_0)$ to $\mathfrak{M}(\lambda_0)$ with $\|F(\lambda)\| = 1$ (for definitions and notation see the review of the cited paper and the review of an earlier paper [ibid. 67, 611–614 (1949); these Rev. 11, 186]), and where $A_{F(\lambda)}$ is the quasi-selfadjoint extension of A generated by $F(\lambda)$ [cf. Livshitz, Mat. Sbornik. N.S. 19(61), 239–262 (1946); these Rev. 8, 588]. Conversely, every $F(\lambda)$ with these properties defines a generalized resolvent R_λ of A, and distinct $F(\lambda)$ define distinct R_λ. The generalized resolvent of A is completely characterized as a family of bounded linear operators R_λ defined on all of $\mathfrak{H}$ and depending on the complex parameter λ in the lower or upper half-plane π and such that for all $h \epsilon \mathfrak{H}$ (1) $(A^* - \lambda I)R_\lambda h = h$, (2) $\|(A^* - \bar{\lambda} I)R_\lambda h\| \leqq \|h\|$, (3) $R_\lambda^* = R_{\bar{\lambda}}$, (4) R_λ is a regular operator function for λ in the half-plane π.

B. Crabtree (Durham, N. H.).

Pallu de la Barrière, Robert. Décomposition des opérateurs non bornés dans les sommes continues d'espaces de Hilbert. C. R. Acad. Sci. Paris 232, 2071–2073 (1951).

Let $H = \int H(\zeta)d\mu(\zeta)$ be a continuous sum of Hilbert spaces in the sense of Godement's [Ann. of Math. (2) 53, 68–124 (1951); these Rev. 12, 421] modification of von Neumann's definition [ibid. 50, 401–485 (1949); these Rev. 10, 548]. The author extends the notion of measurability for operator-valued functions $\zeta \rightarrow A(\zeta)$ to the case in which the $A(\zeta)$ are closed but not necessarily bounded and states three theorems. The first two generalize to the case at hand results about the connection between operator-valued functions and "decomposable" operators in $\int H(\zeta)d\mu(\zeta)$. The third asserts that an H-system in the sense of Ambrose [Trans. Amer. Math. Soc. 65, 27–48 (1949); these Rev. 10, 429] which is of "finite class" is a continuous sum of irreducible such algebras.

G. W. Mackey.

Pallu de la Barrière, Robert. Algèbres auto-adjointes faiblement fermées et algèbres hilbertiennes de classe finie. C. R. Acad. Sci. Paris 232, 1994–1995 (1951).

Thirteen theorems on rings of operators are announced. The first nine of these generalize (to rings which are not necessarily factors) portions of the second paper of Murray and von Neumann [Trans. Amer. Math. Soc. 41, 208–248 (1937)]. The final four are concerned with the connection with the H-systems of Ambrose [ibid. 65, 27–48 (1949); these Rev. 10, 429].

I. Kaplansky (Chicago, Ill.).

Nakano, Hidegorô. Hilbert algebras. Tôhoku Math. J. (2) 2, 4–23 (1950).

A Hilbert algebra is defined to be an algebra A over the complex numbers together with a norm on A in which A is a not necessarily complete Hilbert space, such that for each a in A there is an "adjoint" a^* in A for which left and right multiplication by a and a^* are adjoint operators, and such that left and right multiplication by a is continuous. The basic example of such an A is the set of square integrable functions of compact support on a locally compact unimodular group, with convolution for multiplication. A is called: maximal, if it is contained in no larger Hilbert algebra in the completion H of A; closed, if $a_n \rightarrow h$, with left multiplications by the a_n uniformly bounded, implies $h \epsilon A$; bounded, if multiplication is continuous in both factors. A maximal A is closed. A closed A is bounded if and only if every self adjoint idempotent e is expressible as a finite sum of minimal self-adjoint idempotents. Other similar results are obtained.

W. Ambrose (Cambridge, Mass.).

Utz, W. R. Note on Martin's ergodic function. Amer. Math. Monthly 57, 674–676 (1950).

The author proves two theorems on the ergodic function $\Lambda(\epsilon)$ [cf. the reviewer, Amer. J. Math. 58, 727–734 (1936)], the first of which states $\Lambda(\epsilon)$ is strictly monotone decreasing and that actually $\Lambda(a) - \Lambda(b) \geqq b - a > 0$. If $\rho > 0$ is the radius of the smallest circle containing a bounded set M in the plane he shows that $\rho - \epsilon \leqq \Lambda(\epsilon) < \pi\rho(3+\rho/\epsilon)$ for $0 < \epsilon \leqq \rho$. The paper concludes with a number of unsolved problems and conjectures related to the ergodic function.

M. H. Martin (College Park, Md.).

Theory of Probability

✱von Mises, Richard. Wahrscheinlichkeit, Statistik und Wahrheit. Einführung in die neue Wahrscheinlichkeitslehre und ihre Anwendung. 3d ed. Springer-Verlag, Vienna, 1951. ix+278 pp. $4.30.

This is the third, revised edition of a series of six lectures first published in 1928. According to the author, the new edition differs chiefly from the previous ones in the omission of a considerable part of polemical material which he now considers superfluous, and its replacement by additional material. Prominent in the latter is a polemic, which the author apparently does not consider superfluous, against the modern theory of statistics, of which he strongly disapproves. Also as in previous editions, the author strongly maintains the philosophic position, of which he has long been a prominent and able defender, that a rationally based application of probability to the physical world must be founded on the frequency definition of probability. The reader will find this book an interesting and rather vigorous reaffirmation of the author's views.

J. Wolfowitz.

Varoli, Giuseppe. Ancora sulla probabilità. Period. Mat. (4) 29, 1–10 (1951).

A somewhat polemic discussion of the frequency concept of probability and the law of large numbers.

P. R. Halmos (Montevideo).

✱Good, I. J. Probability and the Weighing of Evidence. Charles Griffin & Co., Ltd., London; Hafner Publishing Co., New York, N. Y., 1950. viii+119 pp. $3.00.

This book is concerned with the foundations of the theory of probability and an analysis of scientific inductive reason-

ing. The author produces rather convincing evidence to support the contention that every application of the theory of probability is based on some probability which is subjectively estimated. He develops a subjective theory in which probability is interpreted as a degree of belief depending on the state of mind of the individual doing the believing. For a body of beliefs to be reasonable the degrees of belief are required to possess a partial order which is invariant under changes in state of mind. Moreover when an individual assigns numerical probabilities to his beliefs then the partial order of the degrees of belief must agree with the numerical order of the probabilities.

The author presents a system of six axioms for numerical probabilities. All of the axioms are stated in terms of conditional probabilities. This device avoids the rather artificial procedure by which Kolmogoroff introduces conditional probabilities. Good's axiom 4 insures that no probability is changed when one of the propositions is replaced by another which is logically equivalent to it. This axiom is important in characterizing logical deductive reasoning. The logical equivalence referred to in axiom 4 is defined in terms of a strict implication. It would have been easier for the reader if some of the properties of this implication had been explicitly stated. In fact implication could be reasonably defined so that axioms 4 and 6 would be consequences of the other axioms. The author shows how the usual results of probability and statistics can be derived from his axioms. He does so in such a manner that a person not trained in probability and statistics can read the book and obtain a fair understanding of those subjects. Many of the proofs are omitted.

The author introduces a concept called the weight of the evidence for a given hypothesis. The weight of evidence is the logarithm of the ratio of the likelihood (R. A. Fisher) of the hypothesis to the likelihood of its negation. The weight may be positive or negative and is additive with respect to independent pieces of evidence. The hypothesis possesses an initial plausibility or weight which is computed in terms of an a priori probability usually subjectively estimated. The final plausibility is obtained by adding the weight of evidence to the initial plausibility. Many of the procedures in statistics can be described in terms of the concepts of plausibility and weight of evidence. This book is interesting reading whether or not one agrees with the author. It is a significant contribution to a field in which contributions are badly needed. *A. H. Copeland, Sr.*

Kappos, Demetrios A. Über die Unabhängigkeit in der Wahrscheinlichkeitstheorie. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1950**, 157–185 (1951).

According to the author the representation of events and random variables in the usual way, as measurable point sets and measurable functions, leads to "Ungereimtheiten". As an alternative basis he proposes the use of Boolean measure rings. [See his earlier paper in the same S.-B. Math.-Nat. Kl. **1948**, 309–320 (1949); these Rev. **11**, 443]. In the present paper he develops the concepts of independent events and random variables from this point of view. *J. L. Doob.*

Zahlen, Jean-Pierre. Sur une application de la logistique à un problème de calcul des probabilités. Euclides, Madrid **10**, 160–165 (1950).

The author obtains a formula for the probability that, of n events, at least k but at most k' will succeed. The result is an immediate consequence of a formula obtained by the reviewer [Amer. Math. Monthly **44**, 213–218 (1937)].

A. H. Copeland, Sr. (Ann Arbor, Mich.).

Rapoport, Anatol. The probability distribution of distinct hits on closely packed targets. Bull. Math. Biophys. **13**, 133–138 (1951).

Any shot results in a hit on exactly one of N possible targets, all targets having equal probabilities. If m targets have already been hit and s shots are fired thereafter, $r_k(s, m)$ is the probability that exactly k new targets will be hit on the volley. The author performs an induction on k to derive

$$r_k(s, m)=\frac{(N-m)!}{(N-m-k)!k!N^s}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(m+k-j)^s.$$

A. S. Householder (Oak Ridge, Tenn.).

Aumann, Georg. Über eine Ungleichung der Wahrscheinlichkeitsrechnung. Acta Univ. Szeged. Sect. Sci. Math. **13**, 163–168 (1950).

The author obtains an inequality relating the variance with the Pasch probability density of a distribution. The inequality is obtained by applying the Schwarz inequality after a transformation of the integrals involved.

A. H. Copeland, Sr. (Ann Arbor, Mich.).

Dieulefait, C. E. On quadratic forms in random variables. An. Soc. Ci. Argentina **4**, 167–172 (1951). (Spanish)

Some elementary distributions associated with the multivariate normal distribution are derived by means of the moment generating function. The moment generating function, and thence the semi-invariants, of an arbitrary quadratic form in independent variables normal about zero with unit variance is computed. From a result which is not particularly obvious without Cochran's theorem, Cochran's theorem is deduced by moment generating functions.

L. J. Savage (Paris).

Dugué, D. Analycité et convexité des fonctions caractéristiques. Ann. Inst. H. Poincaré **12**, 45–56 (1951).

Let f be a function regular in a plane domain D containing an interval I of the real axis, and suppose that (*) $|f(x+iy)|<|f(x)|$, if x is in I and if $y\neq 0$. Then it is proved (**) that for x in I, $f(x)\neq 0$, $\arg f(x)$ is independent of x, and $\log|f(x)|$ is a convex function of x. Conversely if (**) is true, there is a domain containing I in which (*) is satisfied. If for each value of a parameter t, $f(t, x+iy)$ is regular for $x+iy$ in D, and if $f(t, x)$ is a real and convex function of x in I, then $\log \int e^{f(t,x)}dF(t)$ is a convex function of x, for any monotone nondecreasing F. These theorems are applied to prove a variety of results in the theory of probability, including the following. If X is a nonnegative integer-valued random variable, and if $f(x)=E\{x^X\}$, it is proved that $(d/dx)[x/(x-1)]\log f(x)>0$, $0<x\leqq 1$. If P is a polynomial, with $P(1)=1$, and if the expansion of e^P has only positive coefficients, then P has at most one root between 0 and 1. Elegant proofs are obtained of Cramér's [Raikov's] theorem that if a sum of two mutually independent random variables is normal [Poisson] then the summands are also normal [Poisson]. If f is a characteristic function of a random variable, and if f is an entire function of finite order, with an exceptional value in the Borel sense, then f is of order at most 2. If the exceptional value is even exceptional in the Picard sense, the function must be the characteristic function of a (possibly degenerate) normal distribution. *J. L. Doob* (Urbana, Ill.).

Fréchet, Maurice. La moyenne réduite converge "légalement" mais non "en probabilité." Ann. Univ. Lyon. Sect. A. (3) 13, 33–36 (1950).

Let $x_1, x_2, \cdots$ be mutually independent random variables with a common distribution function having a finite positive variance. Then if $a=E\{x_1\}$, the distribution of $s_n=n^{-\frac{1}{2}}\sum_1^n(x_j-a)$ is known to be asymptotically normal. The author remarks that since $s_n-s_{n^2}$ is asymptotically normal with a positive variance, and therefore does not converge to 0 in probability, the sequence $s_1, s_2, \cdots$ cannot converge in probability. *J. L. Doob* (Urbana, Ill.).

Fréchet, Maurice. Généralisations de la loi de probabilité de Laplace. Ann. Inst. H. Poincaré 12, 1–29 (1951).

Two generalizations of normal variables are given. (1) Instead of a numerically valued random variable X consider one with range in a Banach space B. Every linear functional Y on B then defines a numerically valued random variable YX; if the distribution of YX is normal for all functionals Y, then X is called normal. One can consider the characteristic functions $\Phi_Y(X)$. Their totality determines the distribution of X at least whenever B has a denumerable base. (2) Let Σ be a complete metric space and X a random variable with range in it. For every fixed $p\epsilon\Sigma$ the expectation of $(X, p)^2$ is well defined (possibly infinite). Its lower bound for all p is called the variance of X. The author now calls X normal if it has finite variance and if there exists another random variable Y with range in Σ such that Y is independent of X and $X+Y$ is independent of $X-Y$. This definition is suggested by the theorem of S. Bernstein according to which these properties characterize normal variables on the straight line. The author here gives a new proof of this theorem under slightly less restrictive conditions. Finally, it is shown that the two definitions are equivalent in all cases where both apply. *W. Feller* (Princeton, N. J.).

Darmois, Georges. Sur une propriété caractéristique de la loi de probabilité de Laplace. C. R. Acad. Sci. Paris 232, 1999–2000 (1951).

An alternative proof, using characteristic functions, is given for the theorem of S. Bernstein mentioned in the preceding review. The condition of finite variance can be omitted and $X+Y$ and $X-Y$ are replaced by two linear forms. *W. Feller* (Princeton, N. J.).

Dynkin, E. B. Necessary and sufficient statistics for a family of probability distributions. Uspehi Matem. Nauk (N.S.) 6, no. 1(41), 68–90 (1951). (Russian)

An exposition (including a heuristic introduction and detailed proofs) of the author's theory of necessary and sufficient statistics [Doklady Akad. Nauk SSSR (N.S.) 75, 161–164 (1950); these Rev. 12, 427]. *P. R. Halmos.*

★Gnedenko, B. V., and Kolmogorov, A. N. Predel'nye raspredeleniya dlya summ nezavisimyh slučaĭnyh veličin. [Limit Distributions for Sums of Independent Random Variables]. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1949. 264 pp.

This book is based on courses given by the authors at Moscow and Lvov universities. The rate of progress in the theory of probability is illustrated by the fact that almost all of the results in the book have been obtained in the last 15 years (principally by Doeblin, Esseen, Feller, Khintchine, Lévy, and the authors). Furthermore, this book is more than twice as long as Khintchine's older book on the same subject [Limit Theorems for Sums of Independent Random Variables, GONTI, Moscow-Leningrad, 1938].

The first three chapters comprise an introductory section. Chapter I: Probability distributions, random variables, and mathematical expectations. Since only sums of independent random variables are considered, a deep analysis is unnecessary in this chapter. In fact all the results of the book could be stated in terms of distributions and their convolutions, without mentioning measure spaces. However the authors give a brief survey of the measure theoretic foundations of probability, omitting proofs, and making an interesting (but unnecessary, as it seems to the reviewer) restriction on the basic measure space. Probability theory is defined as based on an abstract space of points w, on which a measure is defined, with maximum value 1, which satisfies the following condition. If x is any numerically valued measurable function, that is a random variable, then if A is a linear set, and if $B\supset A$ is open, $\Pr\{x(w)\epsilon A\}=\inf_B \Pr\{x(w)\epsilon B\}$, for every set A for which the left side of the equality is defined. This hypothesis has certain advantages in studying the map of w space on the line, defined by $w\rightarrow x(w)$. The condition is always satisfied, for example, if w space is a complete separable metric space and if the given probability measure is a completed measure of Borel sets. Chapter II: Distributions in R^1 and their characteristic functions. (Proofs are included.) Chapter III: Infinitely divisible distributions. Derivation of the Lévy formula for the characteristic function of such a distribution, with a few heuristic remarks on processes with independent increments.

Chapters IV–VI comprise a second section: General limit theorems. Chapter IV: General limit theorems for sums with independent summands. Concept of (asymptotically) infinitely small summands; conditions necessary and sufficient that their sums, when properly centered and scaled have some, or a given, (infinitely divisible) limiting distribution. Chapter V: Convergence to the normal, Poisson, and unit distributions. Chapter VI: Limit theorems for cumulative sums (that is for partial sums of a single infinite series). There is a section on unimodular distributions, giving results available as yet only in Russian; for example, the theorem that all limiting laws in Lévy's class L (including all stable laws) are unimodular.

Chapters VII–IX comprise a third section: Summands with a common distribution function. Chapter VII: Principal limit theorems (includes a detailed discussion of stable laws). Chapter VIII: Convergence to the normal law. Noteworthy are the asymptotic evaluations of the error in the central limit theorem, and the proof of the central limit theorem for densities of the approximating distributions. Chapter IX: Local limit theorems for the case of lattice distributions.

This book is an invaluable compendium of the most important work on the subject, and is the more striking because of the general lack of systematic and rigorous texts in probability theory. *J. L. Doob* (Urbana, Ill.).

Dvoretzky, A., and Wolfowitz, J. Sums of random integers reduced modulo m. Duke Math. J. 18, 501–507 (1951).

Let X_n be a sequence of mutually independent random variables assuming the values $0, 1, \cdots, m$ with

$$\Pr[X_n=j]=p_n(j).$$

Let Y_n be the sum $X_1+\cdots+X_n$ reduced modulo m. The authors give a very simple proof that Y_n is asymptotically equidistributed if and only if

$$\prod_{n=1}^{\infty}\sum_{j=0}^{m-1}p_n(j)\exp[2\pi irj/m]=0$$

for $r=1, 2, \cdots, m-1$. From this simpler sufficient conditions are derived. Special attention is paid to the case of "essential" equidistribution, that is, when the limiting form is not due to the accident that a certain Y_n (and hence all Y_k with $k>n$) is equidistributed. An estimate of the rapidity of the approach to the limiting form is given.

W. Feller (Princeton, N. J.).

Lévy, Paul. Processus de Markoff. Cas dénombrable. C. R. Acad. Sci. Paris 232, 1803–1805 (1951).

This is a continuation of earlier papers on the same subject [same C. R. 231, 467–468, 1208–1210 (1950); 232, 1400–1402 (1951); these Rev. 12, 269, 619, 723]. Proofs are omitted. In the present paper the author modifies slightly a classification of the processes under consideration as given in his preceding paper. Type V has the property that a sample function H may satisfy the equality $H(t)=h$ on a measurable set E_h of positive measure. If T_t is the measure of the part of E_h in $(0, t)$, $t-T_t$ is the tth random variable of a process with stationary independent increments. Type VI is characterized by nonmeasurable sets E_h. The most general process of this type is described. A type VII is now added to the classification, characterized by nonmeasurable transition probability functions. Examples of such processes have been given by the reviewer [Trans. Amer. Math. Soc. 52, 37–64 (1942); these Rev. 4, 17]. *J. L. Doob.*

Ledermann, Walter. Corrigendum to the paper "On the asymptotic probability distribution for certain Markoff processes." Proc. Cambridge Philos. Soc. 47, 626 (1951).

Cf. same Proc. 46, 581–594 (1950); these Rev. 12, 269.

Nagabhushanam, K. The primary process of a smoothing relation. Ark. Mat. 1, 421–488 (1951).

The author considers the system of equations

$$(*)\qquad L[x(t)]=\sum_0^n a_r x(t-r)=\xi(t),\quad t=0, \pm 1, \cdots,$$

where $\xi(t)$ is the tth random variable of a stationary (wide sense) stochastic process, $a_0, \cdots, a_n$ are given, and the system is to be solved for a stationary (wide sense) $x(t)$ process. [For earlier work on this problem see H. Wold, A Study in the Analysis of Stationary Time Series, Uppsala 1938; Skand. Aktuarietidskr. 21, 208–217 (1938).] Suppose for simplicity that $E\{\xi(t)\}=0$, let σ be the spectral distribution function of the $\xi(t)$ process, and let $P(z)=\sum_0^n a_r z^r$. It is shown that the system (*) has a solution if and only if $\int d\sigma(\lambda)/|P(e^{-i\lambda})|^2<\infty$, where the integration is over the set Q where the denominator does not vanish. The solution is essentially unique if the denominator does not vanish, alternatively if and only if the closed linear manifold of random variables generated by the $x(t)$'s is the same as that (M) generated by the $\xi(t)$'s. Since $L[\xi(0)e^{it\mu}]=0$, if $P(e^{-i\mu})=0$, it follows that the solution is not unique if Q is not empty. [Incidentally this example also shows the incorrectness of Theorem 2.7 according to which there is a unique solution giving random variables in M. There is however a unique solution with spectral density confined to the complement of Q.] If the polynomial P has roots only of modulus $\neq 1$, $x(t)$ has a representation of the form $\sum_{-\infty}^{\infty} b_j\xi(t-j)$. If there are only roots of modulus <1, $b_j=0$ for $j<0$. If there are roots of modulus 1 the above results are modified by replacing simple summation by a summability procedure. The continuous parameter analogues of these results is also obtained.

Assuming (*), now without knowledge of $E\{\xi(t)\}$, but knowing $a_0, \cdots, a_n$ and the covariance function of the $\xi(t)$ process, the problem of estimating $E\{x(t)\}$ from a sample of $\xi(t), \cdots, \xi(t-k+1)$ is taken up. The minimum variance linear unbiased estimate is found. The best least squares approximation to $x(t)$ by means of a linear combination $\xi(t)-\sum_1^q b_j\xi(t-j)$ is found. The corresponding continuous problems, as well as the relation of these problems to linear filtering and prediction are also discussed. *J. L. Doob.*

Pennanéac'h, F. Fonctions aléatoires. Cahiers Rhodaniens 2, 20 pp. (1950).

Expository lecture. *J. L. Doob* (Urbana, Ill.).

Mandelbrot, Benoît. Adaptation du message à la ligne de transmission. II. Interprétations physiques. C. R. Acad. Sci. Paris 232, 2003–2005 (1951).

[For part I see the same vol., 1638–1640 (1951); these Rev. 12, 727.] The analogy between concepts in information theory and quantum mechanics is followed through in detail: word↔pure state; message↔mixture; word probabilities↔point spectrum; cost↔entropy. *J. L. Doob.*

Kosten, L. On the accuracy of measurements of probabilities of delay and of expected times of delay in telecommunication systems. I. Estimates of probabilities of delay. Appl. Sci. Research B. 2, 108–130 (1951).

For simple telephone systems with waiting facility expressions are obtained for the first and second moment of the distributions of the number of delayed calls and of the total time during which all lines are occupied. The nonrandomness of delayed calls proves to have a much greater influence than that of lost calls in the corresponding case without waiting facility. *Author's summary.*

Pollaczek, Félix. Répartition des délais d'attente des avions arrivant à un aéroport qui possède s pistes d'atterrissage. C. R. Acad. Sci. Paris 232, 1901–1903 (1951).

Pollaczek, Félix. Répartition des délais d'attente quantifiés des avions arrivant à un aéroport. II. C. R. Acad. Sci. Paris 232, 2286–2288 (1951).

For part I see the preceding title.

Mathematical Statistics

Fréchet, Maurice. Sur un essai infondé de sauver le coefficient classique dit de corrélation. Rev. Inst. Internat. Statistique 18, 157–160 (1950).

The author gives examples of pairs of random variables such that each has a normal distribution but that their joint distribution is not normal. Moreover, the regression lines are not linear. Their correlation coefficient may be zero even though they are dependent. This neatly refutes a naive, but extremely widespread, belief that the normal correlation theory can be saved for general random variables simply by introducing new scales for each variable. *W. Feller.*

Whitney, D. R. A bivariate extension of the U statistic. Ann. Math. Statistics 22, 274–282 (1951).

Let x, y, and z be three variables with continuous cumulative distribution functions f, g, and h. The problem is to test the hypothesis that $f=g=h$ with the alternative that $f>g$, $f>h$, or $f>g>h$. As a test the author proposes to use

the two statistics U and V where if there are l x's, m y's, and n z's in a sample, the sample values are arranged in ascending order and U counts the number of times a y precedes an x and V counts the number of times a z precedes an x. Critical regions are established for the test. Recurrence relations are given for determining the probability of a given (U, V) in a sample, and for determining the different moments of the joint distribution of U and V. The means, second, and fourth moments are given explicitly and the limit distribution, $l, m, n \to \infty$, is shown to be normal. Numerical examples illustrate the theory. *L. A. Aroian.*

Nagabhushanam, K. Linear transformations and the product-moment matrix. Ann. Math. Statistics **22**, 302–304 (1951).

The author's summary is as follows: "Using linear transformations Rasch has deduced Wishart's distribution [same Ann. **19**, 262–266 (1948); these Rev. **9**, 600]. This note is of the nature of some observations on the Jacobian of the transformation induced by a linear transformation of coordinates with constant coefficients in the distinct elements of the product-moment matrix of a sample of n vectors, each of k components, drawn from a universe of a normal k-variate distribution with zero means." *L. A. Aroian.*

Cole, Randal H. Relations between moments of order statistics. Ann. Math. Statistics **22**, 308–310 (1951).

Let $x_{i|n}$ denote the ith order statistic in decreasing order from a random sample of n taken from a population with a probability density function $f(x)=F'(x)$ where $F(x)$ is the cumulative distribution function. Then the probability distribution function of $x_{i|n}$ is $n\binom{n-1}{i-1}g_{i|n}(x)$ where

$$g_{i|n}(x)=F^{i-1}(x)[1-F(x)]^{n-i}f(x).$$

Let $\nu_{r|n}=\int_{-\infty}^{\infty}g_{r|n}(x)x^t dx$ be called the normalized tth moment of $x_{i|n}$; then $\nu_{r|n-1}=\nu_{r|n}+\nu_{r+1|n}$ since

$$y^{r-1}(1-y)^{n-1-r}=y^{r-1}(1-y)^{n-r}+y^r(1-y)^{n-1-r}.$$

More general relations are found; in particular, it is shown that all normalized moments $\nu_{n|r}$ may be obtained by differencing from $\nu_{1|1}, \nu_{1|2}, \nu_{1|3}, \cdots$ and from these the ordinary moments can be obtained. *M. A. Woodbury.*

Graf, Ulrich, und Henning, Hans-Joachim. Eine Reliefdarstellung der Fisherschen F-Verteilung. Mitteilungsblatt Math. Statist. **3**, 30 (1 plate) (1951).

Evans, W. Duane. On the variance of estimates of the standard deviation and variance. J. Amer. Statist. Assoc. **46**, 220–224 (1951).

Krishna Iyer, P. V. Runs up and down on a lattice. Nature **166**, 276 (1950).

The author obtains the mean and variance for the number of runs up and down on an n-dimensional lattice.

A. H. Copeland, Sr. (Ann Arbor, Mich.).

Cohen, A. C., Jr. Estimation of parameters in truncated Pearson frequency distributions. Ann. Math. Statistics **22**, 256–265 (1951).

Given that the universe sampled obeys a frequency function belonging to the Pearson system but that the sample consists only of the observations following within a limited segment of the range, the author considers the problem of estimating the parameters in the frequency function from such a sample. He assumes that the truncation points (end points of the segment) are known but the number of observations falling outside the segment, and not included in the sample, is unknown. He derives the equations connecting the moments over the truncated range with the parameters, and by substituting the sample moments in these equations he observes that the solutions of the linear equations so obtained give consistent estimates of the parameters. He includes the simpler case when the range is truncated only at one end, also notes the special cases of normal and type III universes. He gives a numerical example in type III case.

C. C. Craig (Ann Arbor, Mich.).

Cohen, A. C., Jr. Estimating parameters of logarithmic-normal distributions by maximum likelihood. J. Amer. Statist. Assoc. **46**, 206–212 (1951).

The author obtains maximum likelihood equations for estimating the three unknown parameters α, β, γ of the logarithmic-normal distribution

$$f(x)=(\gamma x-\alpha)^{-1}(2\pi)^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}\gamma^{-2}\log^2((x-\alpha)/\beta)\}, \quad x>\alpha,$$

and outlines an iterative method for solving these. An alternate estimate for the unknown terminus α based on the smallest sample value is also studied. Asymptotic variances and covariances for the maximum likelihood estimates are found using the information matrix. The case where the terminus α is known a priori is briefly considered.

R. P. Peterson (Seattle, Wash.).

Moran, P. A. P. A curvilinear ranking test. J. Roy. Statist. Soc. Ser. B. **12**, 292–295 (1950).

The minimum number of interchanges D required to change a given permutation of the first n natural numbers to one in which the numbers monotonically increase up to a certain point and then decrease is called the deficit. The author obtains recurrence relations for the distributions of D and tables the entire distributions of D for $n=1(1)8$ and for $D=1(1)12$ for $n=9(1)14$. The probability generating functions for D are given and are used to obtain the mean and variance. These are useful for n large since the distribution of D is asymptotically normal. *M. A. Woodbury.*

Fraser, D. A. S., and Wormleighton, R. Nonparametric estimation. IV. Ann. Math. Statistics **22**, 294–298 (1951).

The paper deals with the problem of constructing tolerance regions on the basis of a sample from a discontinuous distribution; it contains mainly certain corrections to a paper (no. III) with the same title by Tukey [same Ann. **19**, 30–39 (1948); these Rev. **9**, 453]. A peculiar feature of the discontinuous case is that the distributions of the coverages are also discontinuous and, hence, no choice of the tolerance region will exactly meet given confidence level requirements. The original contribution of the present paper consists in certain bounds for the confidence level.

G. Elfving (Helsingfors).

Massey, Frank J., Jr. A note on a two sample test. Ann. Math. Statistics **22**, 304–306 (1951).

If the observations x_{ij} $(i=1, 2, \cdots, p; j=1, 2, \cdots, n_i)$ of p samples from the same continuous population are ordered $z_1, z_2, \cdots, z_N$ $(N=\sum n_i)$, and if $k-1$ of these, $z_{a_1}, z_{a_2}, \cdots, z_{a_{k-1}}$, are determined by previously and arbitrarily chosen subscripts, then the numbers m_{ir} $(r=1, 2, \cdots, k)$ of observations $x_{i\alpha}$ for fixed i such that $z_{a_{r-1}}<x_{i\alpha}\leq z_{a_r}$ $(z_{a_0}=-\infty, z_{a_k}=+\infty)$ have the ordinary contingency table distribution. This result provides a p-sample nonparametric

test of the null hypothesis that p populations all have the same distribution. *A. M. Mood* (Santa Monica, Calif.).

Bennett, B. M. Note on a solution of the generalized Behrens-Fisher problem. Ann. Inst. Statist. Math., Tokyo 2, 87–90 (1951).

The author obtains an exact Student t^2 test for the generalized Behrens-Fisher problem (i.e. that of testing the equality of the means of two multivariate normal populations with unknown covariance matrices Σ' and Σ'') for the case where $\Sigma' \neq \Sigma''$. The test is closely related to that given by Hotelling [Ann. Math. Statistics 2, 360–378 (1931)] for the case where $\Sigma' = \Sigma''$. *R. P. Peterson.*

Walsh, John E. Some nonparametric tests of whether the largest observations of a set are too large or too small. Ann. Math. Statistics 21, 583–592 (1950).

An observation is drawn from each of n populations. Each distribution is assumed to be continuous and symmetric about its median. An inequality among the r largest observations, the r smallest observations, and one other order statistic is given for deciding whether the r largest observations come from populations with medians larger than the other $n-r$ populations with common median. If all populations have common median, the probability of deciding the r largest observations are too large approaches a preassigned number α, and for no admissable n does this probability exceed 2α. Another inequality is recommended for deciding whether the r largest observations are from populations with medians smaller than the other $n-r$ populations. Theorem 4 pertaining to the power of the tests is stated in terms of conditional probabilities, given that the observations from r populations with greater medians are larger than the other observations. The conclusions of theorem 4 concerning the second test mentioned above are incorrect. *T. W. Anderson* (New York, N. Y.).

Isaacson, Stanley L. On the theory of unbiased tests of simple statistical hypotheses specifying the values of two or more parameters. Ann. Math. Statistics 22, 217–234 (1951).

For testing the simple hypothesis that several parameters have specified values, the test is proposed (type D) which among all unbiased tests maximizes the Gaussian curvature of the power surface at the hypothetical point. It is shown that this definition is invariant under suitably regular transformations in the parameter space. Sufficient conditions are given for a test to be of type D, and their application is illustrated with a simple example. *E. L. Lehmann.*

Lyapunov, A. A. On choosing from a finite number of distribution laws. Uspehi Matem. Nauk (N.S.) 6, no. 1(41), 178–186 (1951). (Russian)

Let $F_1, \cdots, F_n$ be probability measures defined on a Borel field of sets of an abstract space X. The author considers the problem of distinguishing which of these F_j's has produced a sample x. The space X is to be expressed as the union of disjunct sets $E_1, \cdots, E_n$, and the statistical rule is to be adopted that F_j is accepted if $x \epsilon E_j$. The reliability of the rule is defined as $\min_i F_i(E_i)$. Suppose that each F_j is nonatomic and absolutely continuous with respect to every other F_i. Then it is shown that there are sets $E_1, \cdots, E_n$ maximizing the reliability. These sets are unique (neglecting sets of F_k-measure 0) if no two F_j's are proportional on the subsets of a set of positive F_k-measure. In all cases $F_i(E_i)$ is independent of i, and this condition is actually sufficient to determine the E_i's in the following very special case. Let $\phi(x, a)$ be for each a a probability density in the real variable x, and suppose that $(\partial^2/\partial a \partial x) \log \phi(x, a) > 0$. Then for any $a_1 < \cdots < a_n$, if F_i above is identified with the distribution determined by the density with $a = a_i$, and if $E_1, \cdots, E_n$ are as above, $E_1, \cdots, E_n$ are intervals, ordered from left to right. The hypothesis on ϕ is satisfied in many cases, for example if ϕ is the normal density with fixed variance and mean a. *J. L. Doob* (Urbana, Ill.).

Anderson, T. W. Classification by multivariate analysis. Psychometrika 16, 31–50 (1951).

An exposition of the fundamental principles involved in choosing among a finite number of populations, given observations on an individual assumed to come from one of the populations. The treatment is based on Wald's approach and the Neyman-Pearson fundamental lemma, covering the Bayes solutions when a priori probabilities are given among populations, and the minimax solutions in the contrary case. Special cases of normal populations are considered, and a numerical example is given for three normal populations. *G. W. Brown* (Santa Monica, Calif.).

Keeping, E. S. A significance test for exponential regression. Ann. Math. Statistics 22, 180–198 (1951).

The paper deals with a method suggested by Hotelling [Amer. J. Math. 61, 440–460 (1939)] for testing the significance of nonlinear regression. Assume that y_i is normally distributed with mean be^{px_i} and variance independent of x_i, $i=1, \cdots, n$. In order to minimize $\sum(y_i - be^{px_i})^2$ we pick the value of p which minimizes the angle between the line through $(y_1, \cdots, y_n)$ and that through $(e^{px_1}, \cdots, e^{px_n})$ in n-dimensional space. Under the null hypothesis that $b=0$, the projection of $(y_1, \cdots, y_n)$ on the unit hypersphere is uniformly distributed. The probability that

$$R = \sum y_i \hat{Y}_i / [\sum y_i^2 \sum \hat{Y}_i^2]^{\frac{1}{2}}$$

exceeds R_0, where the $\hat{Y}_i$ are the fitted values, is, under the null hypothesis, the ratio A/B, where B is the area of the unit hypersphere and A is the area of the tube of points at a geodesic distance $\leqq \cos^{-1} R_0$ from the projection on the unit hypersphere of the curve $Y_i = e^{px_i}$, where p is the curve parameter. The author evaluates the length of this curve on the hypersphere and uses a result of Hotelling to go from this to the area A. A short table of significance levels is given. The case $Y_i = a + be^{px_i}$ is treated by similar methods. It is stated that under the null hypothesis the joint density of the y_i is a function of $\sum x_i^2$; this is a misprint for $\sum y_i^2$. *T. E. Harris* (Santa Monica, Calif.).

Lindley, D. V. A regression problem. Proc. Cambridge Philos. Soc. 47, 337–346 (1951).

The author considers the regression problem proposed by Frisch: If ξ, u, and v are random variables, with $x = a\xi + u$, $y = b\xi + v$, under what conditions is the regression of y on x linear for all a and b? Under the additional assumption that u and ξ are independent, the problem has been answered by E. Fix [Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 79–91, University of California Press, Berkeley-Los Angeles, 1949; these Rev. 10, 553]. The author rederives Fix's results and generalizes them to the case where ξ and u are not necessarily independent but do have finite variances. A two-dimensional variable (ξ', η') is said to have a circular distribution if

arctan (η'/ξ') is rectangularly distributed independently of the distance from the origin; a two-dimensional variable is said to have an elliptical distribution if the variable is a linear transform of a circularly distributed variable. Then there is linear regression in Frisch's problem if and only if ξ and u have a joint elliptical distribution.

K. J. Arrow (Stanford University, Calif.).

Vajda, S. Relations between variously defined effects and interactions in analysis of variance. Ann. Math. Statistics 22, 283–288 (1951).

The author states two conditions upon a nonsingular square matrix, in terms of given positive numbers as weights, for the matrix to be an "h-matrix". The first condition is a general orthogonal condition with respect to the weights, and the second concerns relative constancy. Instances of such matrices arise in the computation of the minimum of a quadratic form, for tests of effects and of interactions in the analysis of variance. The author shows that two h-matrices with the same weights lead to the same minimum, whatever the aggregate of interactions tested, and establishes a condition upon blocks of indices, that the minimum be independent of the h-matrix. *A. A. Bennett.*

Shrikhande, S. S. Designs for two-way elimination of heterogeneity. Ann. Math. Statistics 22, 235–247 (1951).

Numerous new experimental designs are given for dealing with two sources of variation as do Latin squares and Youden squares. The designs are derived from incomplete block designs. One class of designs is obtained by writing the blocks of a balanced incomplete block design in adjacent columns and then permuting the treatments in each column so that every treatment appears the same number of times in every row. It is shown that this is possible for all cases in which the number of blocks is a multiple of the number of treatments and in which the number of replications of a treatment does not exceed ten. Another class of designs is similarly obtained using balanced incomplete block designs in which the number of blocks is not a multiple of the number of treatments. Here there are two classes of treatment comparisons as is the case with the lattice designs. Designs are possible for all instances in which the number of replications does not exceed ten. By similar methods two additional classes of designs are derived from partially balanced incomplete block designs. *A. M. Mood.*

Féron, Robert, et Fourgeaud, Claude. Information et régression. C. R. Acad. Sci. Paris 232, 1636–1638 (1951).

Let X, Y be random variables, and let Y_X be for each x a random variable with distribution the conditional distribution of Y for $X=x$. A natural condition on I_Y, the information relative to Y, is that $E\{I_{Y_X}\}\geqq I_Y$. This condition is always satisfied if, when Y has density function B, I_Y is defined as $\int H[B(y)]dy$, where H is convex. One can write the difference $E\{I_{Y_X}\}-I_Y$ as a sum $E+D$, where E is the gain in "elastic" information and D in "hard" information, corresponding to the concepts of elastic and hard correlation due to Bernstein [Metron 7, no. 2, 3–27 (1927)]. The quantity D is evaluated when the bivariate density of X, Y is a function of a second degree polynomial in its two variables.

J. L. Doob (Urbana, Ill.).

Mathematical Biology

Solomonoff, Ray, and Rapoport, Anatol. Connectivity of random nets. Bull. Math. Biophys. 13, 107–117 (1951).

Given N neurons, each with a axones, each axone synapsing with uniform probability upon any one of the other neurons. If a neuron is selected at random, γ is the expected number of neurons such that to each there is at least one neural path (of one or more elements) from the first. This number γ is called the weak connectivity of the net of neurons and is the quantity sought. The problem has significance, with proper interpretation, for epidemiology and genetics, as well as for nerve net theory. Neglecting variance throughout, the authors define $x(t)$ as the expected number of neurons, to each of which a chain of not more than t axones leads from a given neuron. If $y(t)=N-x(t)$, then $y(t+1)(1-1/N)^{ax(t)}$ is found to be independent of t; hence for N large γ satisfies approximately $\gamma=1-\exp(-a\gamma)$.

A. S. Householder (Oak Ridge, Tenn.).

Rapoport, Anatol. Nets with distance bias. Bull. Math. Biophys. 13, 85–91 (1951).

As a generalization of the results of Solomonoff and Rapoport, the author supposes a linear array of neurons and a probability density $f(r)$, where $f(|x-x_0|)\Delta x$ is the probability that an axone of a neuron at x_0 will synapse with a neuron in the interval Δx about x; $P(t)$ is the probability that a given neuron will be connected to another given neuron by a chain of t elements but not by one of fewer elements; $P(x, t)\Delta x$ is the probability that a neuron at x_0 is connected by a chain of t, but not by one of fewer, elements to a neuron in the interval Δx about x. The author derives a recursion formula for $P(t)$, and one for $P(x, t)$, the latter on the assumption that the entire segment can be subdivided into intervals Δx within each of which the variation of $P(x, t)$ is negligible. *A. S. Householder.*

Sheppard, C. W., and Householder, A. S. The mathematical basis of the interpretation of tracer experiments in closed steady-state systems. J. Appl. Phys. 22, 510–520 (1951).

Similarity and contrast is pointed out between mixing, or "interfusion", of isotopic species with each other and ordinary diffusion. The general theory of interfusion of several species in several compartments is derived and specialized to two configurations of particular interest: several compartments each in communication with a common one, and several compartments in chain. The first special configuration is extended to an infinite analogue. The methods are for the most part those of the theory of linear differential equations with constant coefficients. The work is strongly motivated by biological experimentation. *L. J. Savage.*

Landau, H. G. On dominance relations and the structure of animal societies. I. Effect of inherent characteristics. Bull. Math. Biophys. 13, 1–19 (1951).

The dominance relation is binary, asymmetric and nontransitive. A structure matrix (a_{ij}) is defined where $a_{ii}=0$, and for $i\neq j$, $a_{ij}=1=-a_{ji}$ if i dominates j. The score of i is v_i, the sum of positive a_{ij}; the set of integers v_i is the score structure. The hierarchy index is

$$h=12\sum[v_i-\tfrac{1}{2}(n-1)]^2/(n^3-n),$$

ranging from 0, when all v_i are equal, to 1 for a perfect hierarchy. This is the quantity under investigation. Supposing that p_{ij}, the probability that i dominates j, is a

known function $p(x_i, x_j)$ of "ability vectors" x_i and x_j of the two individuals, the mean and variance of h is obtained in terms of p and the distribution of the ability vectors. Special cases are considered, and it is argued that observed values of h are too high to be accounted for on this simple theory (which takes no account of "social factors, such as social lag, or the effect of existing differences in social rank"). [To the reviewer, this introduction of the index h appears as a significant step forward in the development of the sociological theory of dominance.] *A. S. Householder.*

Wright, Sewall. The genetical structure of populations. Ann. Eugenics 15, 323–354 (1951).

This is the Galton lecture given at University College, London, 1950, supplemented with mathematical appendices. The author's method of path coefficients is applied systematically to problems of population structure under the headings (1) random mating and inbreeding, (2) statistical properties of populations, (3) the inbreeding coefficient F, (4) hierarchic structure, (5) natural populations, (6) the island model of structure, (7) isolation by distance, (8) population structure in evolution, (9) ecologic opportunity, (10) evolution in general; appendices (A) the method of path coefficients, (B) general coefficients of inbreeding, (C) properties of populations as related to F, (D) the inbreeding coefficient of breeds, (E) regular systems of mating, (F) isolation by distance. Quantitative results obtained agree in all comparable cases with, and in other cases supplement, those obtained by various authors otherwise. Thus under (3): "The matrix method [of Bartlett and Haldane] gives a rather complete account of the history of the population . . . . The method of path coefficients yields only one property F, but one can obtain this readily from systems which would require" enormous matrices; and in (7) gene frequency correlations which can be obtained as ratios of inbreeding coefficients seem to agree numerically with a formula (involving Bessel functions) due to Malécot [Les mathématiques de l'hérédité, Masson, Paris, 1948, p. 60; these Rev. 10, 314]. Qualitative conclusions from the analysis support the author's views on evolution.

I. M. H. Etherington (Edinburgh).

Mathematical Economics

✶Karlin, Samuel. Operator treatment of minmax principle. Contributions to the Theory of Games, pp. 133–154. Annals of Mathematics Studies, no. 24. Princeton University Press, Princeton, N. J., 1950. $3.00.

The fundamental equality of the theory of games

$$(*)\quad \inf_g \sup_f \int_0^1\int_0^1 K(x,y)df(x)dg(y) = \sup_f \inf_g \int_0^1\int_0^1 K(x,y)df(x)dg(y)$$

is studied. Here f and g range over all probability distribution functions on $[0, 1]$ while K is a fixed bounded and measurable function (in both variables as well as for each one in the other) in $0\leq x, y\leq 1$. The main tool is the consideration of the operator $Tg=\int K(x, y)dg(y)$. Among other results it is shown that (*) holds whenever the space of Tg is bicompact in the weak topology induced by $(f, Tg)=\int\int K(x, y)df(x)dg(y)$ with f ranging over those distribution functions which are step functions with a finite number of steps. Conditions under which the inf and sup in (*) can be replaced by min and max are studied and applied.

Various generalizations are considered and the validity of the analogue of (*) is shown in more general cases than those hitherto considered [cf. A. Wald, Statistical Decision Functions, Wiley, New York, 1950; these Rev. 12, 193]. The substitution of finitely additive set functions instead of the countably additive distributions is also considered. The value of the paper is, unfortunately, greatly impaired by the extreme laxity with which it is written.

A. Dvoretzky (Ithaca, N. Y.).

✶Bohnenblust, H. F., and Karlin, S. On a theorem of Ville. Contributions to the Theory of Games, pp. 155–160. Annals of Mathematics Studies, no. 24. Princeton University Press, Princeton, N. J., 1950. $3.00.

A theorem of J. Ville [É. Borel, Traité du calcul des probabilités et de ses applications, tome iv, fascicule ii, pp. 105–113, Gauthier-Villars, Paris, 1938] and a fixed point theorem of S. Kakutani [Duke Math. J. 8, 457–459 (1941); these Rev. 3, 60] have proved useful in the theory of games. This paper generalizes the above theorems to Banach spaces. The generalization of Ville's result is preceded by a study of regularly convex sets. The following is a generalization of Kakutani's theorem: With each point x of a convex closed set S in a Banach space let there be associated a nonvoid set $A(x)\subset S$. If $x_n\to x$, $y_n\to y$, $y_n\epsilon A(x_n)$ imply $y\epsilon A(x)$, and if $\bigcup_{x\epsilon S}A(x)$ is contained in a sequentially compact set, then $x_0\epsilon A(x_0)$ for at least one $x_0\epsilon S$. *A. Dvoretzky.*

Sherman, Seymour. Games and sub-games. Proc. Amer. Math. Soc. 2, 186–187 (1951).

Let B be the matrix of game and suppose B is partitioned into submatrices B_j^i of m_i and n_j columns. Suppose the values v_j^i and optimal strategies for the subgames corresponding to the B_j^i are known. Let $\bar{B}$ be the matrix (v_j^i), and let $\bar{v}$ be the value of the corresponding game. Then under suitable conditions information about the solutions of B can be obtained from the solutions of $\bar{B}$. In particular, conditions are given such that the value of B shall be $\bar{v}$. Equivalent results have also been obtained by Gale, Kuhn, and Tucker [Contributions to the Theory of Games, pp. 89–96, Princeton University Press, Princeton, N. J., 1950; these Rev. 12, 514]. *D. Gale* (Providence, R. I.).

Klein, Lawrence R. Stock and flow analysis in economics. Econometrica 18, 236–241 (1950).

This article is part of a controversy inspired by Fellner and Somers [Review of Economics and Statistics 31, 145–146 (1949)], in which it was held that stock and flow analyses were equivalent or could be made equivalent by proper mathematical transformations. Applied to interest theory this implies an equivalence between the liquidity-preference and the loanable-funds theories. The author attempts to show that there exist models in which the choice between stock and flow variables is essential. There is first constructed a simple aggregative model of classical economics which is held to formalize the Fellner-Somers model. This model is static and assumed consistent. It is shown that a unique set of market variables (wage rate, prices, and the rate of interest) is consistent with both the liquidity-preference and the loanable-funds theory of interest. This result is empty. If now the equilibrium conditions contained in the static model are replaced by dynamic excess-demand functions, there can be either a liquidity-

preference or a loanable-funds theory, depending on the choice of equlibrium conditions, but not both. The difference between stock and flow analysis can be expressed in terms of the excess-demand functions chosen for the model. Denote the average price of any economic quantity by p and the supply and demand by y^S and y^D. An adjustment to a flow equilibrium can be represented by the relation $P_t - P_{t-1} = F(y_t^S - y_t^D)$, $F(0)=0$ and an adjustment to stock equilibrium by $P_t - P_{t-1} = F^*[\sum_{i=-\infty}^{t}(y_i^S - y_i^D)]$, $F^*(0)=0$. These relations are equivalent if it is assumed, as it is by Fellner and Somers [loc. cit.], that $\sum_{i=-\infty}^{t-1}(y_i^S - y_i^D)=0$. Whether it is irrelevant to distinguish stock and flow variables is then a question of the usefulness of the models which are assumed. *M. P. Stolts* (Providence, R. I.).

TOPOLOGY

Ore, Oystein. A problem regarding the tracing of graphs. Elemente der Math. 6, 49–53 (1951).

Let a be a vertex of a finite graph G. Let P be any path in G starting from a and constructed according to the rule that from any vertex it is continued along some edge not already traversed by the path and that the path terminates when this ceases to be possible. If every such path traces each edge of G just once and terminates at a the author calls the graph arbitrarily tracable from a. For G to be arbitrarily tracable from a it is of course necessary that G shall be connected and that the degree of each of its vertices shall be even. The author shows that G is then arbitrarily tracable from a if and only if each circuit in G passes through a, that is the graph obtained from G by suppressing a and its incident edges is a forest. [The author's use of the term "tree" is incorrect since connection is not required.]

W. T. Tutte (Toronto, Ont.).

Sprague, R. Über die eindeutige Bestimmbarkeit der Elemente einer endlichen Menge durch zweifache Einteilung. Math. Z. 54, 27–33 (1951).

Let M be a set of n elements. Let E_1 and E_2 be partitions of M into disjoint nonnull subsets. Let us call the pair $\{E_1, E_2\}$ a double partition of M. The author calls it half-descriptive if no permutation of the elements of M other than the identity leaves both E_1 and E_2 unaltered, and fully descriptive if in addition no such permutation merely interchanges E_1 and E_2. The author shows that a fully descriptive double partition of M can be found when $n>5$. The author calls $\{E_1, E_2\}$ "regular" if all the classes of E_1 and E_2 have the same number of elements. He gives a method whereby arbitrarily many examples of regular fully descriptive double partitions of sets may be constructed. He reduces his problem to one in graph theory by regarding the classes of E_1 and E_2 as the vertices of an even graph. Two classes are joined, as vertices of the graph, by one edge for each of their common elements. The double partition is fully descriptive when the group of the graph has no element other than its unity element. *W. T. Tutte.*

Smirnov, Yu. A necessary and sufficient condition for metrizability of a topological space. Doklady Akad. Nauk SSSR (N.S.) 77, 197–200 (1951). (Russian)

The following condition, much simpler than the conditions of P. S. Alexandroff and P. Urysohn [C. R. Acad. Sci. Paris 177, 1274–1276 (1923)], E. W. Chittenden [Bull. Amer. Math. Soc. 33, 13–34 (1927)], and J. W. Tukey [Convergence and Uniformity in Topology, Princeton University Press, 1940; these Rev. 2, 67], is necessary and sufficient for a regular space to be metrizable: There exists a countable collection of locally finite open coverings making up an open basis together. There are two additional results concerning metrizability. *M. Katětov* (Prague).

Wallace, A. D. Extensional invariance. Trans. Amer. Math. Soc. 70, 97–102 (1951).

Let X be a normal T_1 space, and $\beta(X)$ its compactification [cf. E. Čech, Ann. of Math. (2) 38, 823–844 (1937)]. A property P of normal T_1 spaces is "extensional" if $\beta(X)$ has it whenever X has it, "intensional" if the reverse implication holds, and "hereditary" if, whenever X has it, so does every closed subset of X. The author's principal results are these. The following properties are both extensional and intensional: (i) connectedness; (ii) property S (every finite open covering can be refined by one with connected sets); (iii) unicoherence and property S; (iv) dimensional type S, where S is a compact ANR (every continuous mapping of a closed subset of the space in S can be extended to one of the whole space). Further, if P is extensional, intensional and hereditary, then so is the property of being dimensioned by P (every closed subset has arbitrarily small neighborhoods whose boundaries have property P). The last two properties extend results of Hemmingsen [Duke Math. J. 13, 495–504 (1946); these Rev. 8, 334] and Vedenissoff [Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] 5, 211–216 (1941); these Rev. 3, 58]. There are several misprints on pp. 98 and 100; in particular, on p. 98, line 18, the words "if U is open in X'" should be supplied. *A. H. Stone.*

Keldyš, Lyudmila. A continuous transformation of a segment onto an n-dimensional cube. Mat. Sbornik N.S. 28(70), 407–430 (1951). (Russian)

Proofs of results announced earlier [Doklady Akad. Nauk SSSR (N.S.) 66, 327–330 (1949); these Rev. 11, 45].

L. Zippin (Flushing, N. Y.).

Boltyanskiĭ, V. On dimensional full-valuedness of compacta. Amer. Math. Soc. Translation no. 48, pp. 7–11 (1951).

Translated from Doklady Akad. Nauk SSSR (N.S.) 67, 773–776 (1949); these Rev. 11, 195.

Boltyanskiĭ, V. An example of a two-dimensional compactum whose topological square is three-dimensional. Amer. Math. Soc. Translation no. 48, pp. 3–6 (1951).

Translated from Doklady Akad. Nauk SSSR (N.S.) 67, 597–599 (1949); these Rev. 11, 45.

Stein, S. K. Convex maps. Proc. Amer. Math. Soc. 2, 464–466 (1951).

The author defines a planar map M as a partition of the plane by simple arcs and curves into a finite number of connected sets, just one of which is unbounded, such that the union of the bounded regions is homeomorphic to a disc. He investigates the condition that such a map shall be topologically equivalent to a map in which all the bounded regions are convex. He shows that M is equivalent to such a map if its regions are simply connected and each nonnull intersection of two regions is connected. *W. T. Tutte.*

Fort, M. K., Jr. A characterization of plane light open mappings. Proc. Amer. Math. Soc. 2, 175–177 (1951).

Une transformation continue f de plan à plan est dite fonction minimale sur un domaine jordanien S^*, limité par S, si toute fonction continue g sur S^*, coïncidant avec f sur S, est telle que $f(S^*)\subset g(S^*)$. L'auteur démontre qu'une condition nécessaire et suffisante pour qu'une transformation zéro-dimensionnelle soit ouverte (transformation intérieure au sens du référant) est qu'elle soit une fonction minimale sur tout domaine jordanien compris dans le domaine de définition de la transformation. La démonstration fait usage de la notion classique d'indice topologique de la transformation par rapport à la courbe S et à un point fixe $P\in f(S)$, ainsi que des propriétés connues des transformations intérieures. *S. Stoilow* (Bucarest).

Floyd, E. E. Some retraction properties of the orbit decomposition spaces of periodic maps. Amer. J. Math. 73, 363–367 (1951).

Let T be a continuous periodic automorphism of period p of a finite-dimensional locally compact metric space X. The author shows that certain connectivity properties are transmitted from X to X^* where X^* is the space of orbits. These include the property of local connectedness in the sense of homology over I (the integers), local connectedness, local simple connectedness, the property of being an absolute neighborhood retract, that of being an absolute retract. Although the proofs are based on methods which hitherto have only yielded results about homologies over I_p (integers modulo p) where p is a prime, the present theorems are independent of p and can therefore be expected to hold for composite p. The author shows in fact that T and its powers can be replaced by any finite solvable transformation group. *P. A. Smith* (New York, N. Y.).

Toso, Annamaria. A proposito di alcuni teoremi di Trevisan e v. Kerékjártó. Rend. Sem. Mat. Univ. Padova 20, 224–231 (1951).

Let t be an orientation-preserving bicontinuous automorphism of the plane, admitting no fixed points. A simple arc λ with ends A, B is called a translation arc if $B=tA$ and if $\lambda\cap t(\lambda)$ contains no point which is interior to both λ and $t(\lambda)$. Using a fixed-point theorem of Kérékjártó [Acta Litt. Sci. Szeged 4, 86–102 (1928)] the author shows that if λ is a translation arc, the arcs $\lambda+t(\lambda)+\cdots+t^n(\lambda)$, $n=1, 2, \cdots$ are simple. *P. A. Smith* (New York, N. Y.).

Èl'sgol'c, L. È. The variation of the topological structure of level surfaces. Amer. Math. Soc. Translation no. 47, 27 pp. (1951).

Translated from Mat. Sbornik N.S. 23(65), 399–418 (1948); these Rev. 10, 392.

Čogošvili, G. S. On level surfaces and regions of smaller values. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 17, 203–243 (1949). (Russian. Georgian summary)

According to the preface, the substance of the paper was presented in Lyusternik's Moscow seminar in 1938–1939 and published in part in four short notes [C. R. (Doklady) Acad. Sci. URSS (N.S.) 22, 293–297 (1939); 24, 635–639 (1939); Bull. Acad. Sci. Georgian SSR [Soobščenia Akad. Nauk. Gruzinskoĭ SSR] 3, 995–999 (1942); 4, 853–859 (1943); these Rev. 1, 320; 5, 214; 6, 165]. The author considers the level sets $[f=a]$ and the smaller value sets $[f\leqq a]$ of a function F defined on an n-dimensional manifold, subject to certain customary assumptions concerning differentiability and nondegeneracy, with the object of obtaining, among other things, the critical point relations under general boundary conditions of Morse and van Schaak [Ann. of Math. (2) 35, 545–571 (1934)]. The main theorems on which the author's discussion is based are of the same type as those of his third note, referred to above, and somewhat similar to work (which he quotes) by Èl'sgol'c [see, for instance, Mat. Sbornik N.S. 23(65), 399–418 (1948); these Rev. 10, 392]. They concern modifications of level sets $[f=a]$, or of smaller value sets $[f\leqq a]$, which remain homeomorphic as the number a passes through a critical value of f or of its boundary function. *L. C. Young.*

Čogošvili, G. S. On homological approximations and laws of duality for arbitrary sets. Mat. Sbornik N.S. 28(70), 89–118 (1951). (Russian)

This contains the detailed proofs of earlier notes [C. R. (Doklady) Acad. Sci. URSS (N.S.) 46, 131–132 (1945); C. R. Acad. Sci. Paris 221, 15–17 (1945); these Rev. 7, 37, 216]. Some of the author's work was used and proofs given by P. Alexandroff [Mat. Sbornik N.S. 21(63), 161–232 (1947); these Rev. 9, 456]. The author's results are not restricted to finite-dimensional sets and are more general than those of Alexandroff [loc. cit.] and of Kaplan [Trans. Amer. Math. Soc. 62, 248–271 (1947); these Rev. 9, 456].

The author defines interior and exterior homology and cohomology groups of subsets A and B of a locally bicompact normal space R. For the principal results R has vanishing homology in dimensions r and $r+1$, and A and B are complementary: $B=R/A$. In the succeeding formulas, summarizing the duality relations, X is a discrete coefficient group and θ its bicompact character group. In each formula the first exterior group is isomorphic to the second and both are dual, in the author's sense, to the third group:

$$\Delta_e^{r+1}(B,\theta)\approx\Delta_i^r(A,\theta)\|\nabla_e^{r+1}(B,X),$$
$$\nabla_e^{r+1}(B,X)\approx\nabla_i^r(A,X)\|\Delta_e^{r+1}(B,\theta).$$

All the author's homology-groups are limit groups, of direct or indirect spectra, based on the totality of locally-finite coverings of R by open sets. The generalized duality of groups L and M implies that M, say, is isomorphic to a uniquely described everywhere dense subset of the character group of L and that L is the group of all continuous characters of M. *L. Zippin* (Flushing, N. Y.).

Supnick, Fred. On the perspective deformation of polyhedra. II. Solution of the convexity problem. Ann. of Math. (2) 53, 551–555 (1951).

It is shown that there exist geodesic triangulations of a sphere (in particular, of octahedral type) which cannot be carried by central projections into convex polyhedra. With the aid of linear inequalities, such triangulations are characterized, and an algorithm produced for finding all convex polyhedra which are central projections of a given geodesic triangulation. *S. S. Cairns* (Urbana, Ill.).

Cockcroft, W. H. Note on a theorem by J. H. C. Whitehead. Quart. J. Math., Oxford Ser. (2) 2, 159–160 (1951).

For a connected CW-complex K, with n-section K^n, J. H. C. Whitehead [Bull. Amer. Math. Soc. 55, 453–496 (1949); these Rev. 11, 48] proved that the group $\rho_2=\pi_2(K^2, K^1)$ is a free crossed $(\pi_1(K^1), d)$-module, where d is the natural map of $\pi_2(K^2, K^1)$ into $\pi_1(K^1)$. The author now gives a new and simpler proof for this, which avoids Whitehead's geometrical

arguments, and uses instead the natural homomorphism of ρ_2 onto the group of 2-chains of the universal covering $\tilde{K}$ and the fact that the latter group is a free $\pi_1(K^2)$-module.

H. Samelson (Ann Arbor, Mich.).

***Hopf, H. Introduction à la théorie des espaces fibrés.** Colloque de topologie (espaces fibrés), Bruxelles, 1950, pp. 9–14. Georges Thone, Liège; Masson et Cie., Paris, 1951. 175 Belgian francs; 1225 French francs.

In this expository paper, the author gives an intuitive description of the concept of a fibre bundle and amplifies his description by means of some examples. There is also a discussion of some of the principal problems of the theory, and some of the major results that have been achieved.

W. S. Massey (Providence, R. I.).

Whitehead, George W. A generalization of the Hopf invariant. Ann. of Math. (2) 51, 192–237 (1950).

Den Gegenstand der Untersuchung bilden die Homotopiegruppen der Sphären, $\pi_n(S^r)$; die Hauptfrage lautet: Für welche (n, r) ist $\pi_n(S^r)\neq 0$, d.h. für welche (n, r) gibt es wesentliche Abbildungen $S^n\to S^r$? Bis zum Erscheinen der vorliegenden Arbeit war dies für die folgenden (n, r) bekannt (dabei ist k immer eine beliebige positive ganze Zahl):

$$(k, k);\ (4k-1, 2k);\ (k+2, k+1);\ (k+6, k+3);\ (k+14, k+7);\ (4, 2);\ (8, 4);\ (10, 4);\ (16, 8);\ (18, 8);\ (22, 8).$$

Diese Liste wird jetzt folgendermassen ergänzt:

$$(8k, 4k);\ (8k+1, 4k+1);\ (16k+2, 8k);\ (16k+3, 8k+1);\ (14, 7);\ (14, 4);$$

und zwar wird in jedem Fall eine wesentliche Abbildung explizit angegeben.

Die Bedeutung der Arbeit besteht aber nicht nur in diesen neuen Resultaten, sondern besonders auch in der Methode und in der ganzen Art und Weise, auf welche die Resultate gewonnen werden. Es werden zunächst in äusserst exakter Weise ältere Methoden zur Untersuchung der Gruppen $\pi_n(S^r)$ auseinandergesetzt, analysiert und zueinander in Beziehung gebracht, und zwar werden hauptsächlich die Theorie der Verschlingungsinvariante H_0, die vom Referenten für die Abbildungen $S^{2r-1}\to S^r$ eingeführt worden ist und auf die sich der Titel der Arbeit bezieht, und der Einhängungsprozess von H. Freudenthal behandelt; daneben folgende Operationen in Homotopiegruppen: das Produkt von J. H. C. Whitehead; die Zusammensetzung $S^n\to S^r\to X$, welche jedem Paar $\alpha\epsilon\pi_n(S^r)$, $\beta\epsilon\pi_r(X)$ ein Element von $\pi_n(X)$ zuordnet; die Verbindung ("join") zweier Elemente von $\pi_p(S^r)$ und $\pi_q(S^s)$, die ein Element von $\pi_{p+q+1}(S^{r+s+1})$ ist (für $s=0$ ist dies die Freudenthalsche Einhängung); eine vom Referenten (in einem Spezialfall) benutzte Konstruktion, die jeder Abbildungsklasse $S^p\times S^q\to S^r$ ein Element von $\pi_{p+q+1}(S^{r+1})$ zuordnet. Darauf werden die Homotopiegruppen der Räume $A\vee B$—das sind die Vereinigungen zweier Räume A, B, die genau einen Punkt gemeinsam haben—untersucht; als Spezialfall allgemeinerer Formeln, welche an einen älteren Satz von J. H. C. Whitehead anknüpfen, wird bewiesen: $\pi_n(S^r\vee S^r)\approx\pi_n(S^r)+\pi_n(S^r)+\pi_n(S^{2r-1})$ für $n<3r-3$; diese direkte Summendarstellung ist kanonisch, es ist also durch sie ein Homomorphismus φ der Gruppe $\pi_n(S^r\vee S^r)$ auf ihren direkten Summanden $\pi_n(S^{2r-1})$ gegeben. Nun sei $Q: S^r\to S^r\vee S^r$ die Abbildung, die dadurch entsteht, dass man eine Äquatorsphäre von S^r auf einen Punkt zusammenschnürt, und $f: S^n\to S^r$ eine Abbildung, die das Element $\alpha\epsilon\pi_n(S^r)$ repräsentiert; dann repräsentiert φQf ein Element $H(\alpha)\epsilon\pi_n(S^{2r-1})$, und die Abbildung $H:\pi_n(S^r)\to\pi_n(S^{2r-1})$ ist ein Homomorphismus; in dem Fall $n=2r-1$ kann man die Elemente von $\pi_n(S^{2r-1})$ als ganze Zahlen deuten, und in diesem Falle ist $H(\alpha)$ die alte Invariante H_0; diese wird also durch den Homomorphismus H, der für $n<3r-3$ erklärt ist, verallgemeinert. Noch eine zweite wichtige und ebenfalls garnicht naheliegende Verallgemeinerung wird vorgenommen: in Freudenthals Arbeit [Compositio Math. 5, 299–314 (1937), § 8] werden jeder Nullhomotopie der Einhängung Ef einer Abbildung $f: S^{2r-1}\to S^r$ zwei Zahlen c', c'' zugeordnet, welche zwar in keinem der formulierten Sätze auftreten, aber in den Beweisen eine ausschlaggebende Rolle spielen; jetzt werden für beliebige (n, r) die Nullhomotopien der Einhängungen Ef von Abbildungen $f: S^n\to S^r$ in Homotopieklassen eingeteilt und die Gesamtheit dieser Klassen wird in natürlicher Weise als Gruppe π_{r+1}^{n+2} aufgefasst; zwei Homomorphismen Λ', Λ'' von π_{r+1}^{n+2} in $\pi_{n+2}(S^{2r+1})$ sind ausgezeichnet, die für $n=2r-1$ mit den Freudenthalschen Zahlen c', c'' äquivalent sind. Analog wie bei Freudenthal dienen Λ', Λ'' der feineren Untersuchung des Homomorphismus H, der Einhängung E und der Beziehungen zwischen H und E. Bei Gelegenheit dieser Untersuchungen wird ein Satz von Freudenthal (der sich auf den alten Fall $n=2r-1$ bezieht) verschärft.

Auf Grund der so entwickelten Methoden wird dann auf 3 Seiten die oben reproduzierte neue Liste der Paare (n, r) gewonnen, für welche wesentliche Abbildungen $S^n\to S^r$ existieren. Im Anschluss an die Behandlung des Falles $(8k+1, 4k+1)$ wird noch bewiesen, dass es für $r=4k+2$ keine Abbildungen $S^{2r-1}\to S^r$ mit der Invariante $H_0=1$ gibt; dies ist aus verschiedenen Gründen interessant; hier wird die folgende Konsequenz hervorgehoben: Auf den Sphären der Dimensionen $4k+1$ gibt es keine stetige Multiplikation mit Links- und Rechts-Einselement.

Der Leser wird in dieser ungewöhnlich reichhaltigen Arbeit zahlreiche weitere interessante Einzelheiten finden, auf die ich hier nicht eingegangen bin. [Ein kleines Versehen: auf der ersten Seite, Zeile 5 von unten, sind "odd" und "even" miteinander vertauscht.] *H. Hopf* (Zürich).

Steenrod, N. E., and Whitehead, J. H. C. Vector fields on the n-sphere. Proc. Nat. Acad. Sci. U. S. A. 37, 58–63 (1951).

Les entiers n et k étant reliés par la condition que $(n+1)/2^k$ soit un entier impair, les auteurs démontrent le théorème suivant: Sur la sphère S^n munie de sa structure différentiable naturelle, il n'existe pas 2^k champs de vecteurs tangents, linéairement indépendants en tout point. Ils donnent une série d'énoncés équivalents et de corollaires, avec renvoi au livre récent de Steenrod [The Topology of Fibre Bundles, Princeton University Press, 1951; ces Rev. 12, 522]. Citons notamment l'impossibilité, pour $m>n$, de fibrer S^m par S^n (avec le groupe orthogonal comme groupe structural) si $n+1\neq 2^k$; et l'inexistence de structure presque complexe (subordonnée à la structure différentiable usuelle) sur S^n, pour $n\neq 2^k-2$. La démonstration du théorème principal repose sur la formule $Sq^iu^q=c_{i,q}u^{q+i}$, valable en cohomologie mod 2 pour les puissances d'une classe de cohomologie u de degré 1; $c_{i,q}$ désigne le coefficient du binôme $\binom{q}{i}$ réduit modulo 2. Appliquant ceci à la cohomologie de l'espace projectif réel P^n, on trouve que 2^k est le plus petit des entiers j tels que $Sq^ju^{n-j}\neq 0$; ceci prouve l'impossibilité d'une section dans la variété de Stiefel $V_{n+1,2^k+1}$ de base S^n, d'où résulte le théorème. *H. Cartan* (Paris).

Borel, Armand. Sur la cohomologie des variétés de Stiefel et de certains groupes de Lie. C. R. Acad. Sci. Paris **232**, 1628–1630 (1951).

Results are announced on the homology structure, both additive and multiplicative, of the following spaces: the unimodular orthogonal group in n variables, its universal covering group spin (n), the exceptional simple Lie groups G_2, F_4 in 14 and 52 variables, Stiefel manifolds, and their analogues in complex and quaternion variables. In particular, the cohomology ring mod 2 of spin (n) is determined; moreover, spin (n) has torsion coefficient if and only if $n \geqq 7$. The only torsion coefficient of G_2 is 2, while F_4 has torsion coefficients $\leqq 3$. Whether the latter has torsion $=3$ is undecided. The method is by means of the spectral cohomology theory as applied to fiber bundles. *S. Chern.*

Hodge, W. V. D. A special type of Kähler manifold. Proc. London Math. Soc. (3) **1**, 104–117 (1951).

The Kähler manifolds in question, said to be of restricted type by the author, have the property that the cycle Δ (of dimension $2m-2$ when m is the complex dimension of the manifold) corresponding to the fundamental 2-form in the sense of intersection theory is homologous to a multiple of an integral cycle Γ. The author proved that for such manifolds U_m a base for the ineffective p-cycles of class h can be found consisting of integral cycles. If $z^1, \cdots, z^m$ are the local coordinates on the manifold, a p-form

$$P^{h,k} = \sum_{\alpha_1<\cdots<\alpha_h} \sum_{\beta_1<\cdots<\beta_k} P_{\alpha_1\cdots\alpha_h\beta_1\cdots\beta_k} dz^{\alpha_1}\cdots dz^{\alpha_h} d\bar{z}^{\beta_1}\cdots d\bar{z}^{\beta_k}$$

is said to be of type (h, k). Then any p-form can be written in one and only one way as a sum $P = P^{p,0} + P^{p-1,1} + \cdots + P^{0,p}$. Let $\rho^{h,k}$ denote the number of linearly independent effective $(h+k)$-forms of type (h, k). Let Γ_i, $i=1, \cdots, R_{2p}-R_{2p-2}$ (R_{2p} = Betti number of dimension $2p$) be an integral base for the $(2m-2p)$-cycles of U_m whose intersections with Γ^{m-2p} are effective $2p$-cycles. It is proved that the signature of the intersection matrix $\|I(\Gamma_i, \Gamma_j, \Gamma^{m-2p})\|$ is

$$(\rho^{2p,0}+\rho^{2p-2,2}+\cdots+\rho^{0,2p},\ \rho^{2p-1,1}+\rho^{2p-3,3}+\cdots+\rho^{1,2p-1}).$$

Moreover, the period matrix of the effective integrals of type (1, 0) is a Riemannian matrix. A partial converse of this theorem is given. A necessary condition is then proved for a Kähler manifold to be of restricted type. The paper also contains a theorem on general Kähler manifolds to the effect that the numbers $\rho^{h,k}$ are invariants of the complex structure and independent of the choice of the Kähler metric on it. *S. Chern* (Chicago, Ill.).

Pontryagin, L. S. Some topological invariants of closed Riemannian manifolds. Amer. Math. Soc. Translation no. **49**, 54 pp. (1951).

Translated from Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 125–162 (1949); these Rev. **10**, 727.

Rado, Tibor. A remark on chain-homotopy. Proc. Amer. Math. Soc. **2**, 458–463 (1951).

The author shows that, starting with any finite simplicial complex K having torsion, one can always construct a finite simplicial complex L and a pair of chain maps $K \rightarrow L$ which are integrally homologous but not chain-homotopic. If n is the highest dimension in which torsion appears, $L = K \cup$ cone over n-skeleton of K; the chain maps are slightly more complicated. This result may also be regarded as a construction elaborating a known general theorem [S. Lefschetz, Algebraic Topology, Amer. Math. Soc. Colloq. Publ., v. 27, New York, 1942, (17.10) on p. 157; these Rev. **4**, 84]. *J. Dugundji* (Los Angeles, Calif.).

GEOMETRY

Thébault, Victor. Sur des triangles associés. Ann. Soc. Sci. Bruxelles. Sér. I. **65**, 5–14 (1951).

Thébault, Victor. Sur la géométrie du tétraèdre. Ann. Soc. Sci. Bruxelles. Sér. I. **65**, 49–56 (1951).

Ehrhart, E. Le triangle orienté. Mathesis **60**, 15–23, 92–104 (1951).

If α, β, γ, are three coplanar axes intersecting in $A(\beta, \gamma)$, $B(\gamma, \alpha)$, $C(\alpha, \beta)$ such that ABC is the positive sense of rotation of the oriented plane, the axes are said to form an oriented triangle of base (T_0) whose sides a, b, c, and angles A, B, C, are defined as follows: $a=\overline{BC}$, $b=\overline{CA}$, $c=\overline{AB}$, $A(\gamma, \beta_1)$, $B(\alpha, \gamma_1)$, $C(\beta, \alpha_1)$, taken between 0 and $+\pi$, where α_1, β_1, γ_1 are the axes obtained when α, β, γ, respectively, are rotated about one of their points to make an angle of $+\pi$ with the initial position. The elements of (T_0) are thus identical with the usual elements of the triangle ABC. However, those definitions are still to be applied to the oriented triangle (T) derived from (T_0) by making the three given axes vary in a continuous manner. The angles of (T) may lie outside the interval $(0, \pi)$ and the sides may assume any algebraic values.

Now let the oriented triangle (T_0) be referred to the axes $\overrightarrow{BX}=\alpha$, $\overrightarrow{BY}$ such that $(\overrightarrow{B\alpha}, \overrightarrow{BY}) = +\frac{1}{2}\pi$, and the triangle (T) to the analogous pair of axes. Two points of (T_0), (T) are said to be homologous if they are the same functions of the analogous elements of the two triangles, and similarly for the equations of homologous curves. An examination of the relations between homologous pairs of points and homologous pairs of curves leads the author to the following law of permanence: Any proposition or relation which holds for one position of an oriented triangle whose sides vary in a continuous manner remains valid for any other position of the triangle. Many examples of the fruitfulness of this law are given. Thus it suffices to show that the nine-point circle is tangent to the incircle of a triangle to obtain, by the aid of this law, the complete theorem of Feuerbach. The author is also careful to warn of the weaknesses or limitations of the law. *N. A. Court* (Norman, Okla.).

Jensen, Henry. The six infinite regular polyhedra. Mat. Tidsskr. A. **1950**, 53–60 (1950). (Danish)

A polyhedron is said to be regular if it possesses two particular symmetry operations: one cyclically permuting the vertices of a face, and one cyclically permuting the faces at a vertex. When the number of faces is allowed to be infinite, this criterion admits the simple plane tessellations of triangles, squares, and hexagons, and also three "skew" polyhedra discovered by Petrie and Coxeter in 1926. These have, at each vertex, six squares, four hexagons, and six hexagons, respectively. The present paper provides a clear exposition, with excellent drawings, of the reviewer's proof [Proc. London Math. Soc. (2) **43**, 33–62 (1937)] that these are the only possible regular skew polyhedra in Euclidean 3-space. *H. S. M. Coxeter* (Toronto, Ont.).

Hope, C. The nets of the regular star-faced and star-pointed polyhedra. Math. Gaz. **35**, 8–11 (1 plate) (1951).

The author describes a practical method for constructing paper models of the four regular star polyhedra of Kepler and Poinsot. Unhappily he has interchanged the names of the great dodecahedron and the small stellated dodecahedron. The source of this mistake seems to be the Encyclopaedia Britannica, 14th edition [vol. 20, p. 965, art. "Solids, geometric"]. For the correct usage see the 11th edition [vol. 22, p. 28, art. "Polyhedron"]. *H. S. M. Coxeter.*

Zeckendorf, É. Étude fibonaccienne. Mathesis **60**, supplément, 35 pp. (1951).

Continuing his previous work [Mathesis **58**, 44–49 (1949), 293–306 (1950); these Rev. **11**, 153, 417] the author considers the sections of a regular four-dimensional simplex by a sequence of parallel hyperplanes in a carefully chosen direction, in order to obtain a three-dimensional projection as "irregular" as possible. *H. S. M. Coxeter.*

Motzkin, Th. The lines and planes connecting the points of a finite set. Trans. Amer. Math. Soc. **70**, 451–464 (1951).

The author proves among others the following theorems: (1) Any n points in d-dimensional space that are not on one hyperplane determine at least n connecting hyperplanes. (2) Let there be given n points in the plane not all in a line. Then there exist at least $cn^{\frac{1}{2}}$ lines which go through exactly two of the given points. (The fact that there exists at least one such line is due to Gallai.) The author further conjectures: Any n points in d-dimensional space that are not on one hyperplane determine at least one connecting hyperplane on which all but one of the given points are on a linear $(d-2)$-space. For $d=2$ this is Gallai's theorem. The author proves it for $d=3$. The general case is still unsolved. Several related problems are considered. The author also considers the problem in which the points are situated in more general spaces than the Euclidean ones. In the introduction the history and background of these problems is given.

P. Erdös (Aberdeen).

Buerger, M. J. Vector sets. Acta Cryst. **3**, 87–97, 243 (1950).

A set of n points determines n^2 vectors between pairs of points. Let these n^2 vectors be referred to a common origin; the author shows how to determine, in so far as this is possible, the original n points. He also discusses the connections between the symmetries of the two sets. [The mathematical basis of the applications of the author's results to crystallography have been criticized by Hartman and Wintner, Physical Rev. (2) **81**, 271–273 (1951); these Rev. **12**, 495.] *R. P. Boas, Jr.* (Evanston, Ill.).

Nowacki, Werner. Beziehungen zwischen der Symmetrie des Kristall-, Fourier- und Patterson-Raumes. Schweiz. Mineral. Petrog. Mitt. **30**, 147–160 (1950).

Rajagopal, C. T. On the intersections of a central conic and its principal hyperbolas. Math. Gaz. **35**, 97–104 (1951).

Riabouchinsky, Dimitri. Sur la construction graphique dans le plan cartésien des courbes dites imaginaires, adjointes aux réelles. C. R. Acad. Sci. Paris **232**, 2275–2278 (1951).

Wottling, A. Quelques cas d'homographie sur une quadrique et leurs conséquences planes. Cahiers Rhodaniens **2**, 40 pp. (1950).

Gambier, B., et Hocquenghem, A. Ellipses ayant deux sommets consécutifs donnés. J. Math. Pures Appl. (9) **29**, 275–311 (1950).

In this paper the authors have made a detailed study of the one-parameter family of ellipses (E) which have two fixed points A and B as "consecutive summits", that is, when either A or B is an end of the major axis, the other is an end of the minor axis of an ellipse of (E). In the coordinate system xOy, A has the coordinates $(R, 0)$ and $B(-R, 0)$. The locus of the centers of the ellipses of (E) is a circle of radius R and center O. The envelope of (E) is an elliptic curve $\mathcal{E}$ of order 8 and class 4 consisting of two ovals with the common center O. Each oval is parallel to and at the distance R from an ellipse of eccentricity $\frac{1}{2}\sqrt{2}$ and with vertices at $(\pm 2R, 0)$. One oval $\mathcal{E}_1$ is outside and the other $\mathcal{E}_2$ (containing the points A and B) is inside of this ellipse. The point equation of $\mathcal{E}$ is derived. The curve $\mathcal{E}$ has eight imaginary cusps, four imaginary nodes on the line at infinity, and triple points at A and B, each with a single tangent. In addition to A and B and imaginary intersections, each proper ellipse of (E) has a real contact with $\mathcal{E}$. All of the ellipses (E) are contained in the interior of $\mathcal{E}_1$. There are four ellipses of (E), real or imaginary, through each point or tangent to each line of the plane. The numbers of such real ellipses are obtained for certain points and lines. Two of the ellipses of (E) are circles of radius $R\sqrt{2}$ each of which is rotated about Ox generating a torus whose properties are studied. The locus of the point of contact of two ellipses of (E) and the envelope of their common tangent are obtained. *T. R. Hollcroft* (Aurora, N. Y.).

Fladt, Kuno. Über die Transformationen der Hauptgruppe im Euklidischen Raume. Math.-Phys. Semesterber. **2**, 104–116 (1951).

An exposition of the use of quaternions for describing congruent transformations and dilatations in Euclidean 3-space. *H. S. M. Coxeter* (Toronto, Ont.).

Bilinski, Stanko. Verallgemeinerung eines Satzes von G. Monge. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **5**, 175–177 (1950). (Serbo-Croatian. German summary)

***Cartan, É. Leçons sur la géométrie projective complexe.** 2d ed. Gauthier-Villars, Paris, 1950. vii+325 pp. 1100 francs.

Nouveau tirage of a book first published in 1931.

Hohenberg, Fritz. Logarithmische Spiralen im komplexen Gebiet. Monatsh. Math. **55**, 54–61 (1951).

Laguerre [Oeuvres, v. 2, Gauthier-Villars, Paris, 1905, pp. 88–108] represented the general point P of the complex Euclidean plane by the ordered pair of real points $P_1 = PI \cdot \bar{P}J$, $P_2 = PJ \cdot \bar{P}I$, where I, J are the circular points and $\bar{P}$ is the conjugate of P. Study [Vorlesungen über ausgewählte Gegenstände der Geometrie, v. 1, Teubner, Leipzig-Berlin, 1911, p. 48] observed that an analytic curve determines a transformation $P_1 \rightarrow P_2$ which leaves invariant any real points on the curve. The author considers this transformation in the special case when the curve has the polar equation $r = ce^{k\theta}$, where c and k are complex constants. By stereographic projection he derives some properties of loxodromes on a sphere. *H. S. M. Coxeter.*

***Verriest, Gustave. Introduction à la géométrie non euclidienne par la méthode élémentaire.** Gauthier-Villars, Paris, 1951. viii+193 pp. 1000 Francs.

This book is intended for students who want to acquire the first notions of the non-Euclidean geometry. The larger part of the work is devoted to Lobachevsky's (L.) geometry. The author starts with the familiar set of axioms by Hilbert, but Dedekind's continuity axiom is taken instead of the two axioms Hilbert uses. In the next chapter the author studies various consequences of these axioms. Much care has been taken to distinguish between the theorems which are common to both geometries and those which are not. Thus, in the second chapter only the consequences of the axioms of position and order are discussed, whereas in the third chapter the axioms of congruence are also taken into consideration. In the fourth chapter the continuity axiom is added. In the next chapters the elementary theory of polygons and trapezoids is treated. Chapter VIII deals with the area of the triangle in L. geometry. Chapters IX and X are devoted to the metric properties in L. geometry: the notion of parallelism, distance between two lines, infinite or ideal points, asymptotic triangles, circles, horicircles and hypercircles and finally the representation of L. geometry in Euclidean geometry. The last chapter gives the principles of Riemannian geometry and shows that it is equivalent to the geometry on a sphere or the geometry of a sheaf.

H. A. Lauwerier (Amsterdam).

Böheim, Hermann. Krümmungskreise und Evoluten reeller Kegelschnitte bei Cayley-Klein'scher Metrik. Monatsh. Math. 55, 43–53 (1951).

In der vorliegenden Arbeit hat Verfasser die geläufige Betrachtungen betr. oskulierende Kegelschnitte, Krümmungskreise und Evoluten auf die nicht-Euklidische Geometrie erweitert. Dazu benutzt er eine projektive Ebene welcher durch einen Cayley-Kleinschen Masskegelschnitt eine nicht-Euklidische Metrik aufgeprägt ist.

H. A. Lauwerier (Amsterdam).

Primrose, E. J. F. Quadrics in finite geometries. Proc. Cambridge Philos. Soc. 47, 299–304 (1951).

The author finds the number of nondegenerate quadrics in the P.G.(s, n). If s is even there is only one type of quadric. If s is odd there are two types, ruled and unruled quadrics. The author finds the number of nondegenerate quadrics of each type and the number of points on each type. The quadrics may be used as the blocks in an incomplete balanced block design, the points of the P.G.(s, n) being the varieties.

H. B. Mann (Columbus, Ohio).

Ionesco, D. V. Quelques problèmes de géométrie finie. Acad. Roum. Bull. Sect. Sci. 30, 264–269 (1947).

Let a surface S be represented by $z=f(x, y)$, continuous in x and y, and let $z_0=f(u, v)$, $z_1=f(u+\lambda, v+\mu)$, $z_2=f(u-\lambda, v+\mu)$, $z_3=f(u-\lambda, v-\mu)$, $z_4=f(u+\lambda, v-\mu)$ where u, v is a point of the (x, y)-plane and λ, μ are parameters. Let V denote the volume in the usual sense between $z=f(x, y)$ and $z=0$ within limits $x=u\pm\lambda$, $y=v\pm\mu$. The author states three problems: to determine the respective surfaces S for which the respective ratios $(z_1+z_2+z_3+z_4)/z_0$, V/z_0, $V/(z_1+z_2+z_3+z_4)$ are independent of u and v. The functional equations in the respective cases are reduced to the same single partial difference equation studied by the author in an earlier paper. Solutions of the single equation are enumerated and adapted to each of the stated problems.

P. E. Guenther (Cleveland, Ohio).

Kelly, L. M. Distance sets. Canadian J. Math. 3, 187–194 (1951).

Let S be a metric space. The distance set $D(S)$ is defined as follows: The nonnegative real number x is an element of $D(S)$ if and only if there are two points of S whose distance is x. Zero is always in $D(S)$. The principal result of this paper is that for every n there exists a set of $n+1$ real numbers and 0 which are not distance sets for any subset of the n-dimensional Euclidean space. Some further results are: Any set of n positive numbers and 0 is the distance set of a subset of n-dimensional Euclidean space. Any countable set and 0 is a distance set of a subset of Hilbert space. Several problems are also raised.

P. Erdös (Aberdeen).

Convex Domains, Extremal Problems, Integral Geometry

Vincensini, Paul. Sur les ensembles convexes et les involutions algébriques de directions du plan. C. R. Acad. Sci. Paris 232, 2075–2076 (1951).

Let γ be a set of closed arcs of a circle Γ. A theorem of Horn and Valentine [Duke Math. J. 16, 131–140 (1949); these Rev. 10, 468] asserts that if every two arcs in γ have a common point, then there is a point $x\epsilon\Gamma$ such that every arc in γ contains either x or the point diametrically opposite x. The author proves (by applying Helly's theorem [Jber. Deutsch. Math. Verein. 32, 175–176 (1923)] to their convex hulls) that if every three arcs in γ have a common point 'but no point is in all arcs of γ' then Γ admits an involution such that for each $p\epsilon\Gamma$ and $A\epsilon\gamma$, A contains either p or its conjugate point. He deduces from this an analogous result concerning convex sets of directions, and appends a correction to an earlier paper [Bull. Sci. Math. (2) 59, 163–174 (1935)]. [Reviewer's note: The author states erroneously that if '$\cdots$' above is false, then every involution of Γ has the desired property.]

V. L. Klee, Jr.

Besicovitch, A. S. Measure of asymmetry of convex curves (II): Curves of constant width. J. London Math. Soc. 26, 81–93 (1951).

Let Γ be a closed convex plane curve and let Γ_1 be a central closed curve of maximal area inscribed in Γ; let $|\Gamma|$ be the area of the closed plane figure with boundary Γ. Then $\alpha=|\Gamma_1|/|\Gamma|$ is a measure of the asymmetry of the convex curve Γ. Part I [same J. 23, 237–240 (1948); these Rev. 10, 320] showed that $1\geqq\alpha\geqq\frac{2}{3}$ and that the greatest asymmetry, $\alpha=\frac{2}{3}$, is attained when Γ is a triangle. This paper shows that if Γ is restricted to the class of curves of constant breadth, then the greatest asymmetry (minimum α) is attained when Γ is the Reuleaux triangle; an exact expression for this minimum is given; it is approximately .840 as contrasted with the minimum $\frac{2}{3}$ in the larger class of convex curves.

M. M. Day (Urbana, Ill.).

Freilich, Gerald. On sets of constant width. Proc. Amer. Math. Soc. 2, 92–96 (1951).

Let C be a convex body of constant width d in a Euclidean space. It is shown that the lengths of the segments intercepted by C on each of two parallel lines at most δ distance apart differ by at most $2(2\delta d)^{\frac{1}{2}}$, a value depending on the width but not on the shape of C. Similar results for higher dimensional parallel cross-sections are obtained from this one by integration.

W. Gustin (Princeton, N. J.).

Bateman, Paul, and Erdös, Paul. Geometrical extrema suggested by a lemma of Besicovitch. Amer. Math. Monthly **58**, 306–314 (1951).

Let S be a set of n points in the plane between which the mutual distances are at least 1. Let $r(n)$ be the radius of the least circle having a point of S as center containing S and $D(n)$ the diameter of S. Among many other results it is proved that $D(7)\geqq 2$ and $r(20)>2$. It is shown that the last inequality involves the following theorem suggested by a lemma of Besicovitch [Proc. Cambridge Philos. Soc. **41**, 103–110 (1945); these Rev. **7**, 10]. In a set of a finite number of coplanar circles, the center of no one of them being in the interior of another, the number of circles meeting the least one does not exceed 18. *L. Fejes Tóth* (Veszprém).

Stamm, Otto. Umkehrung eines Satzes von Archimedes über die Kugel. Abh. Math. Sem. Univ. Hamburg **17**, 112–132 (1951).

It is a familiar property of the sphere that the area cut from the surface by a pair of parallel planes is proportional to the distance between the planes. The author here presents two proofs of the converse theorem that if an ovaloid has the property that the area cut off by parallel planes is proportional to the distance between them, then the ovaloid is a sphere. In the statement of the problem, the proportionality factor is allowed to be a function of the normal to the planes. It is noted that the case when the proportionality factor is independent of direction was discussed by Blaschke [Vorlesungen über Differentialgeometrie . . . , Band I, Springer, Berlin, 1921, pp. 83, 161]. It is shown by example that to establish this result it is not sufficient to assume the postulated property only for the planes perpendicular to a fixed plane. *S. B. Jackson* (College Park, Md.).

Hadwiger, H. Beweis eines Funktionalsatzes für konvexe Körper. Abh. Math. Sem. Univ. Hamburg **17**, 69–76 (1951).

In problems on integral geometry in ordinary Euclidean space four invariants of a body have played an important rôle. These are the total curvature, the integral of mean curvature, the surface area, and the volume. Following some earlier work of Blaschke, the author proves that for convex bodies they can be characterized by the properties: (1) invariance under rigid motions; (2) additivity, i.e., $\varphi(A)+\varphi(B)=\varphi(A+B)+\varphi(AB)$; (3) continuity. The theorem will have numerous applications in integral geometry. *S. Chern* (Chicago, Ill.).

Matildi, Pietro. Una precisazione in merito ad un lemma interessante la teoria elementare degli isoperimetri. Atti Relaz. Accad. Pugliese Sci. N.S. **7**, Parte I, 301–316 (1949).

Correction of a mistake in an elementary problem [Enriques, Questioni riguardanti le mathematiche elementari, parte III, Zanichelli, Bologna, 1927, art. XXVI by Chisini, p. 201]. *L. C. Young* (Madison, Wis.).

Algebraic Geometry

Bagchi, Hari Das, and Mukherji, Biswarup. Note on a circular cubic with a real inflexion at infinity. Bull. Calcutta Math. Soc. **42**, 73–81 (1950).

The authors derive certain properties of elliptic and rational circular cubics with a real inflection at infinity, and show that each of these properties may be used to define the cubic with which it is associated. In particular, they obtain the known result that the trisectrix of Maclaurin is the only rational circular cubic with three inflections at infinity. *T. R. Hollcroft* (Aurora, N. Y.).

Rosina, Bellino Antonio. Sul numero dei diametri principali di una curva algebrica piana con punti all'infinito non tutti distinti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10**, 213–216 (1951).

The diameter of a plane algebraic curve of order n conjugate to a given direction being defined as the locus of mean centres of the sets of points traced by lines parallel to that direction, i.e. the polar line of their common point at infinity, and a principal diameter as one that is perpendicular to the direction to which it is conjugate, it is shown that the number d of distinct principal diameters is at most h, the number of distinct points at infinity on the curve; if the terms of orders n, $n-1$ in the Cartesian equation of the curve are divisible by $(x^2+y^2)^r$, $(x^2+y^2)^s$ respectively, where $2r<n$, $s<r-1$, then $d\leqq n-2$. If, however, $2r=n$, then every diameter is principal, and the infinity of principal diameters are concurrent in a point if also $s=r-1$. *P. Du Val.*

Huff, Gerald B. Cremona's equations and the properness inequalities. Duke Math. J. **17**, 385–389 (1950).

The author considers Cremona's equations and the "properness inequalities" for a regular linear system of plane curves of dimension r, order x_0 and multiplicities $x_1, x_2, \cdots, x_n$ at a set of base points $P_1, P_2, \cdots, P_n$, the general curve being irreducible and of genus g, and examines their arithmetic implications. He demonstrates, as it had been already conjectured (for $g=0$, $r>0$) by A. B. Coble [Algebraic geometry and theta functions, Amer. Math. Soc. Colloq. Publ., vol. 10, New York, 1929, pp. 11, 12], that for $g=0, 1$, and $r>0$ all solutions are indeed proper.
M. Piazzola-Beloch (Ferrara).

Gandin, Renato. Sulla determinazione geometrico-funzionale del gruppo dei punti di contatto di un sistema di spazi con una curva algebrica. Rend. Sem. Mat. Univ. Padova **19**, 54–61 (1950).

The solution of the general problem of the multiple secants of an algebraic curve $C_p{}^n$, of order n, genus p and without singular points, in S_r, depends upon determining the S_k of S_r which have ν_i-point contact with $C_p{}^n$. This problem has been solved for a rational curve by Severi. The author solves the following particular case of the general problem by a method due to A. Comessati [Atti Ist. Veneto Sci. Lett. Arti. **69**$_2$, 871–881 (1910)]: To determine the group of ν-point contacts of the S_{r-2} of S_r with a $C_p{}^n$ of S_r, each S_{r-2} having with $C_p{}^n$, in addition to the ν-point contact, a $(r-\nu-2)$-point contact and a further simple intersection.
T. R. Hollcroft (Aurora, N. Y.).

Bydzovský, B. Sur certains points remarquables d'une cubique rationnelle plane. Časopis Pěst. Mat. Fys. **75**, 219–229 (1950). (French. Czech summary)

Without referring to articles on this topic by Winger [Bull. Amer. Math. Soc. **25**, 27–44 (1918)] and Allen [Amer. J. Math. **49**, 456–461 (1927)], the author considers a nodal cubic in the form (1) $x_1{}^3+x_2{}^3-6x_1x_2x_3=0$ and defines a point P of (1) to be $3n$-ponctuel if there is a curve of order n having intersection multiplicity $3n$ with (1) at P. He then duplicates the earlier results which relate configurations of

$3n$-ponctuel points to the covariant conics of (1) (conics touching the nodal tangents at their points of intersection with the line of inflections), and to sets of points which are closed under the tangential process. *G. B. Huff.*

✱Nagell, Trygve. Sur quelques questions dans la théorie arithmétique des cubiques planes du premier genre. Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 59–64. Centre National de la Recherche Scientifique, Paris, 1950.

The present paper summarizes some of the results obtained by Poincaré, Mordell, Weil, Nagell, Billing, Mahler, Wiman and Néron in the arithmetical theory of the plane cubics of genus 1. After having recalled the reduction of the equation of such a cubic to Weierstrass form and the Mordell-Weil theorem on the finite basis, many other results are given concerning the rank, the resolvent fields (especially the quadratic ones) the study of birational and linear equivalence by means of Aronhold invariants, and the exceptional points. It is shown that the Mordell-Weil theorem cannot be extended to arbitrary fields, and that the number of exceptional points is not a birational invariant. *B. Segre* (Rome).

✱Châtelet, François. Points exceptionnels des cubiques. Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 71–72. Centre National de la Recherche Scientifique, Paris, 1950.

By using the methods of E. Lutz [J. Reine Angew. Math. **177**, 238–247 (1937)] the author shows how the problem of rational exceptional points upon an elliptic cubic of genus 1 can be solved in any algebraic field k, by determining the order of these points, i.e. their order in the Abelian group of points of the cubic belonging to k. Also some extensions of the problem above are considered. *B. Segre* (Rome).

✱Néron, A. Les propriétés du rang des courbes algébriques dans les corps de degré de transcendance fini. Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 65–69. Centre National de la Recherche Scientifique, Paris, 1950.

Let k, K', K be three fields such that $K'=k(u_1, \cdots, u_n)$ is a purely transcendental extension of k, and $K=K'(\alpha)$ is an algebraic (simple) extension of K' as well as an extension of finite transcendental degree of the rational field. We obtain an homomorphism σ of K in a field K^σ, by giving certain values $u_1^\sigma, \cdots, u_n^\sigma$, arbitrarily chosen in k, for the variables $u_1, \cdots, u_n$. If Γ is a curve of genus $p\geqq 1$, defined in K, suppose that it has finite rank r in K (i.e., r is the rank of the Abelian group formed by the points of the Jacobi V_r of Γ belonging to K). The author shows that it is then possible to determine σ, in an infinity of different ways, so that the transform Γ^σ of Γ has rank $\geqq r$ in K^σ. Incidentally he proves that the number of points of a curve $\Phi(z, u)=0$ of genus $\geqq 1$ and finite rank r, defined over the rational field, such that z is rational and u is an integer $<N$, has an upper bound $B(\log N)^r$, where B is a constant. He then constructs, over a finite algebraic field, plane elliptic cubic curves of ranks 8, $\geqq 9$, $\geqq 10$, and curves of arbitrary genus $p\geqq 2$ of rank $\geqq 3p+6$ [cf. Néron, C. R. Acad. Sci. Paris **226**, 1781–1783 (1948); **228**, 1087–1089 (1949); these Rev. **10**, 60, 623]. *B. Segre* (Rome).

✱Segre, Beniamino. Sur les points entiers des surfaces cubiques. Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 81–82. Centre National de la Recherche Scientifique, Paris, 1950.

The algebraic surface $F(x, y, z)=0$ has infinitely many integral points (x, y, z) if there is a substitution $x=a_1u+b_1v+c_1$ etc. with integer coefficients such that $f(x, y, z)=\phi(u, v)$ and $\phi(u, v)=0$ has infinitely many integer points, e.g., is a cubic with integer coefficients and a cusp at an integral point. Examples are (i) $hz^3=f(x, y)$, where h is an integer and $f(x, y)$ is an irreducible inhomogeneous quadratic form with integer coefficients such that $f(x, y)=0$ has an integer solution and (ii) $hz^2=f(x, y)$, where h is an integer and $f(x, y)$ is an irreducible inhomogeneous cubic with integral coefficients such that $f(x, y)=0$ has an integral point of inflexion. *J. W. S. Cassels* (Cambridge, England).

✱Segre, Beniamino. Questions arithmétiques sur les variétés algébriques. Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 83–91. Centre National de la Recherche Scientifique, Paris, 1950.

A compact survey of, and an excellent guide to the arithmetical questions which arise when algebraic varieties are considered over any commutative field, not necessarily the field of complex numbers. In this branch of mathematics, which the author has considerably enriched by his own results, the tendency is to produce arithmetical results out of geometrical results. A comprehensive bibliography is appended. *D. Pedoe* (London).

✱Segre, Beniamino. Sur un problème de M. Zariski. Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 135–138. Centre National de la Recherche Scientifique, Paris, 1950.

If V_r, V_r' are two cones of dimension $r\geqq 2$, and V_{r-1}, V'_{r-1} are their generic hyperplane sections, then it is clear that when V_{r-1} and V'_{r-1} are isomorphic (birationally equivalent) V_r and V_r' are also isomorphic. Zariski has raised the converse question: (a) Does an isomorphism between V_r and V_r' ensure that V_{r-1} and V'_{r-1} are likewise isomorphic? This question is given a more restricted formulation: (b) Does each isomorphism between V_r and V_r' transform the generating lines of V_r into those of V_r'? It is observed that when the answer to (b) is affirmative the answer to (a) is also, but the case in which V_r and V_r' are projective spaces shows that an affirmative answer to (a) does not imply an affirmative answer to (b). The varieties V_r for which a negative answer must be given to (b) must contain two distinct congruences Σ, $\bar{\Sigma}$ of curves C, $\bar{C}$ which satisfy the conditions: (1) The generic C ($\bar{C}$) is unicursal; (2) Σ ($\bar{\Sigma}$) is of index 1; (3) V_r contains a subvariety W ($\bar{W}$) unisecant to the curves C ($\bar{C}$). Such varieties are investigated in some detail, and in the case $r=2$ it is shown that they are rational and in the case $r=3$, those which are not rational carry a pencil of rational surfaces. While these cases are exceptions to (b) they are not so to (a), so that in the cases $r=2$, 3, the isomorphism of V_{r-1} and V'_{r-1} is necessary and sufficient for that of V_r and V_r'. *H. T. Muhly* (Iowa City, Iowa).

Segre, Beniamino. Bertini forms and Hessian matrices. J. London Math. Soc. **26**, 164–176 (1951).

Soit V_n^d une variété de dimension d et de classe n de l'espace projectif S_r. Les hyperplans tangents à V forment

en genéral une famille de dimension $r-1$ et de degré n, et satisfont donc à une équation homogène de degré n, $F(u_0, \cdots, u_r)=0$ en les coordonnées (u_i) d'un hyperplan; F est appelée la forme de Bertini de V. Sur un corps de caractéristique $\neq 2$, une condition nécessaire et suffisante pour qu'une forme F de degré n en les u_i soit la forme de Bertini d'une $V_n{}^d$ est que la matrice Hessienne de F (c'est à dire la matrice des dérivées secondes de F) soit de rang $d+2$ mod F. Ainsi, à part l'exception des $V_n{}^d$ développables (dont le cas n'est qu'esquissé ici) et le remplacement du degré par la classe, la forme de Bertini a les mêmes propriétés que la "zugeordnete Form" de Chow pour représenter une variété. L'auteur donne aussi diverses propriétés de divisibilité des Hessiens et matrices Hessiennes de déterminants symétriques et de discriminants, et des applications géométriques de celles ci. *P. Samuel.*

Errera, A. Un problème diophantien de M. Segre. Extrait d'une lettre de M. Siegel. Bull. Soc. Roy. Sci. Liège **19**, 404–405 (1950).

Repairs an omission and simplifies a step in an earlier paper [same Bull. **19**, 177–186, 213–214 (1950); these Rev. **12**, 200]. *J. W. S. Cassels* (Cambridge England).

Matsusaka, Teruhisa. The theorem of Bertini on linear systems in modular fields. Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **26**, 51–62 (1950).

This is a generalization of Zariski's proof [Trans. Amer. Math. Soc. **50**, 48–70 (1941); these Rev. **2**, 345] of Bertini's theorem to include fields of characteristic $p\neq 0$. Let Σ/k be the field of rational functions on a given normal r-dimensional variety V/k; let k be algebraically closed in Σ, and let Σ/k be separably generated. Zariski has used the fields P, $k\subset P\subseteq\Sigma$, $\dim P/k=1$, to define pencils of V_{r-1}'s on V, and the pencil is noncomposite if and only if P is algebraically closed in Σ. In the case of ch. $p\neq 0$, the author adds the condition that P be not contained in $k(\Sigma^p)$, or, what is the same thing, that Σ/P be separably generated. With a similar appropriate modification of the definition of reducible linear system, the theorems given by Zariski continue to hold, word for word. *A. Seidenberg.*

Abellanas, Pedro. Fundamental subvariety for an algebraic correspondence. Revista Mat. Hisp-Amer. (4) **10**, 207–232 (1950). (Spanish and English)

In a previous paper [same Revista (4) **9**, 175–233 (1949); these Rev. **12**, 740] the author has considered an irreducible algebraic correspondence T between two irreducible algebraic varieties V, V'; if a and b are the dimensions of the transforms by T, T^{-1} of the generic points of V, V' respectively, and if $\mathfrak{P}'$ is any irreducible component of $T[\mathfrak{P}]$, where $\mathfrak{P}$ denotes any irreducible subvariety of V, then he has proved that

$$(1)\qquad \dim\mathfrak{P}'\geqq\dim\mathfrak{P}+a-b.$$

Now he shows that the inequality sign must hold in this relation, if $\mathfrak{P}$ is fundamental for T and $\mathfrak{P}'$ is not fundamental for T^{-1}. In particular, when $b=0$ and T^{-1} has no fundamental variety on V', it follows that a necessary and sufficient condition for $\mathfrak{P}$ to be fundamental is $\dim\mathfrak{P}'>\dim\mathfrak{P}+a$. Also the ideal representing a fundamental subvariety $\mathfrak{P}$ of T on V is constructed, whence the relation $\dim\mathfrak{P}\leqq\dim V-2$ is deduced. *B. Segre* (Rome).

Fernandez Biarge, Julio. Coincidences in an algebraic correspondence. Revista Mat. Hisp.-Amer. (4) **10**, 160–170 (1950). (Spanish)

The author has previously studied the coincidence varieties of a birational transformation of an algebraic variety V into itself [same Revista (4) **10**, 3–11 (1950); these Rev. **12**, 529]. Here he extends the investigation to a general algebraic transformation T of V into itself. A coincidence variety of T is a subvariety A of V which is a component of $T(A)$ [and so also of $T^{-1}(A)$]. Under certain regularity hypotheses, the coincidence of T in such a variety A is classified as perfect, imperfect and superior, according to the behaviour of T in the 1st order neighbourhood of A; moreover, certain correspondences between directions in this neighbourhood are investigated. *B. Segre* (Rome).

***van der Waerden, B. L. Les valuations en géométrie algébrique.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 117–121. Centre National de la Recherche Scientifique, Paris, 1950.

An expository article, outlining two applications of valuation theory to algebraic geometry: (a) Zariski's work on the reduction of singularities [Ann. of Math. (2) **40**, 639–689 (1939); **41**, 852–896 (1940); **43**, 583–593 (1942); **45**, 472–542 (1944); these Rev. **1**, 26; **2**, 124; **4**, 52; **6**, 102]; (b) the author's own work on birational invariants of linear systems [Acta Salmanticensia. Ciencias: Sec. Mat., no. 2 (1947); these Rev. **9**, 248]. *I. S. Cohen* (Cambridge, Mass.).

Zariski, Oscar. Theory and applications of holomorphic functions on algebraic varieties over arbitrary ground fields. Mem. Amer. Math. Soc., no. **5**, 90 pp. (1951). \$1.40.

Let V be an irreducible algebraic variety defined over a field k, and let G be a nonempty subset of it. A function ξ^* on V is said to be defined on G if for every point of G we are given an element $\xi_P{}^*$ of the completion of the local ring O_P of P, subject to the condition that if P and Q are k-isomorphic points of G (and hence $O_P=O_Q$), then $\xi_P{}^*=\xi_Q{}^*$. If ξ^* has the property that there exists a sequence $\{\xi_i\}$ of elements of the function field of V such that at each point P of G the elements ξ_i belong to O_P and converge to $\xi_P{}^*$, the convergence being uniform on G, then ξ^* is said to be strongly holomorphic on G. If there exists a finite set of algebraic subvarieties Γ_h $(h=1, \cdots, r)$ of V such that the sets $G-\Gamma_h$ cover G, and if ξ^* is strongly holomorphic in each set $G-\Gamma_h$, then ξ^* is said to be holomorphic on G.

It is clear that if k is the field of complex numbers and V is an affine n-space, this definition of holomorphic functions is equivalent to the usual definition of holomorphic functions of several complex variables, but in its general form it is much wider, since it covers arbitrary ground fields and arbitrary varieties V, which may be in affine or in projective space. The first part of the memoir under review develops the general theory of holomorphic functions on an arbitrary variety. In the affine case it is shown that every holomorphic function is strongly holomorphic, and this, in effect, allows the theory to be developed completely for affine models. In the case of projective models, however, the theory is not complete, and a number of unsolved problems are listed, but the theory is carried far enough for the applications made later.

If T is a birational transformation of V which is regular at every point of G, it is trivial to see that there is a natural isomorphism of the ring of holomorphic functions on G onto

the ring of holomorphic functions on $T\{G\}$. However, if T is a more general transformation, such a result does not necessarily hold, and the second part of the memoir is devoted to an examination of this problem. The most general result is established as follows. Suppose that T^{-1} is a rational transform of V' into V, and let W be an irreducible subvariety of V, $W'=T\{W\}$, the total transform of W by T. If (1) V is locally normal at every point of W, (2) to each point P' of W' there corresponds in T^{-1} a unique point P of V, where $O_P \subset O_{P'}$, (3) the function field of V is maximally algebraic in the function field of V', then there is an isomorphism of the ring of functions on V which are holomorphic on W onto the ring of functions on V' which are holomorphic on W'. This theorem is first proved when T is a birational transformation, and then the proof is extended to give the more general result.

The principal applications of the theory of holomorphic functions on an algebraic variety are to problems dealing with the analytic reducibility of a variety, and to questions of connectedness. An irreducible variety V is analytically irreducible along a subvariety W if the ring of functions on V holomorphic on W is an integral domain. In the case of normal varieties, it is shown that a necessary and sufficient condition for this is that W be connected. Applying the theory of the second section of this memoir, it is proved that if T is an irreducible algebraic correspondence between two irreducible varieties V/k and V'/k, and if W/k is a connected subvariety of V and $W'=T\{W\}$, then W' is also connected, provided that the following conditions are satisfied: (1) If (P, P') is a general point of T/k, $k(P)$ is maximally algebraic in $k(P, P')$; (2) V/k is analytically irreducible at each point of W. This theorem, which even in the classical case is more general than any previously known result, is applied to give a proof, valid in the most general case, of the well-known principle of degeneration of Enriques, the result proved being to the effect that if the general cycle of an irreducible algebraic system is absolutely irreducible, then every cycle of the system is absolutely connected.

W. V. D. Hodge (Cambridge, England).

Zariski, Oscar. A fundamental lemma from the theory of holomorphic functions on an algebraic variety. Ann. Mat. Pura Appl. (4) 29, 187–198 (1949).

[Cf. the preceding review.] Let V be an irreducible algebraic variety over an arbitrary ground field, W an irreducible subvariety, G a subset of W such that W is the smallest algebraic variety containing G. For any point P of V let $m(P)$ denote the ideal of nonunits in the local ring of P on V. The lemma in question then states: If z is a rational function on V and $z\epsilon m(P)^k$ for every point P of G, then also $z\epsilon m(P)^k$ for every general point P of W. The proof involves a detailed consideration of certain groups arising from certain local rings. For the special case where W is a simple subvariety of V, another, and very short, proof is given of the lemma. Easy consequences of the lemma are the following two theorems, the first being its principal application. 1. If W_0 is the set of all algebraic points of W, then the natural homomorphism of the ring of holomorphic functions along W into the ring of holomorphic functions along W_0 is an isomorphism onto. 2. Let R be a nonhomogeneous coordinate ring of V with respect to which W is at finite distance, and let $\mathfrak{P}$ be the prime ideal of W in R. Then for every integer k there exists an integer h such that $\bigcap_{P\epsilon W} m(P)^h \cap R \subset \mathfrak{P}^k$.

I. S. Cohen (Cambridge, Mass.).

Kneser, Hellmuth. Analytische Mannigfaltigkeiten im komplexen projektiven Raum. Math. Nachr. 4, 382–391 (1951).

L'auteur donne une démonstration du théorème de Chow [Amer. J. Math. 71, 893–914 (1949); ces Rev. 11, 389]: Toute sous-variété analytique (complexe) M de l'espace projectif complexe P_n est algébrique. Contrairement à la démonstration de Chow, celle-ci ne fait pas usage de la théorie topologique des intersections. Elle consiste: (1) à décomposer M en ses composantes (globales) des différentes dimensions; ceci nécessite un assez long rappel, sans démonstrations, de propriétés plus ou moins classiques des germes de variétés analytiques ("analytische Mengenkeime"); (2) à démontrer le théorème de Chow d'abord pour une M de dimension $n-1$ (dans ce cas, le théorème était connu avant Chow; il résulte d'un théorème de P. Thullen sur les singularités des sous-variétés analytiques de dimension $n-1$ dans une variété ambiante de dimension n, théorème que d'ailleurs l'auteur ne cite pas [Math. Ann. 111, 137–157 (1935)]); (3) à démontrer le théorème pour une M de dimension $k \leqq n-2$, en prouvant, par récurrence sur n, que M est intersection de cônes algébriques de dimension $k+1$. [Note: Comme le rapporteur l'a indiqué dans une conférence au Congrès International des Mathématiciens à Harvard en 1950, le théorème de Chow est une conséquence immédiate du suivant: Une sous-variété analytique de dimension $k \geqq 1$, dans une variété ambiante de dimension n, ne peut pas avoir de point singulier essentiel isolé.] *H. Cartan* (Paris).

Gröbner, W. Über den idealtheoretischen Beweis des Satzes von Bézout. Monatsh. Math. 55, 82–86 (1951).

Consider n hypersurfaces in projective n-space, of respective equations $F_1=0, \cdots, F_n=0$, where $F_1, \cdots, F_n$ are forms of respective degrees $g_1, \cdots, g_n$. Assume that these hypersurfaces have only a finite number of points in common. If P is any one of these points, there is a primary component $\mathfrak{q}$ of the ideal $\mathfrak{a}=(F_1, \cdots, F_n)$ whose associated prime ideal is the prime ideal which corresponds to P; the author defines the multiplicity of P in the intersection to be the length of the ideal $\mathfrak{q}$. He observes that, with this definition, the proof of Bézout's theorem, viz. that the sum of the multiplicities of the points of intersection is $g_1 \cdots g_n$, follows immediately from a simple property of the Hilbert function of an ideal. The sentence which begins on line 17, p. 83 and which is interrupted on line 22 continues on line 21, p. 84, and the sentence whose first half constitutes the end of the paper continues on line 23, p. 83.

C. Chevalley (New York, N. Y.).

***Samuel, P. Multiplicités d'intersection en géométrie algébrique.** Algèbre et Théorie des Nombres. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 24, pp. 123–124. Centre National de la Recherche Scientifique, Paris, 1950.

This report contains a concise summary of certain of the author's earlier contributions [C. R. Acad. Sci. Paris 225, 1111–1113, 1244–1245 (1947); 228, 158–159 (1949); these Rev. 9, 325; 10, 732] to the theory of intersection multiplicities. *H. T. Muhly* (Iowa City, Iowa).

Barker, C. C. H. Intersections and contact of surfaces on a V_3. J. London Math. Soc. 26, 125–131 (1951).

Gegeben seien zwei algebraische Flächen S und T auf einer Mannigfaltigkeit V_3 und ihre invarianten und kovarianten System der Äquivalenz; sie mögen eine Kurve C gemeinsam haben, deren kanonisches System ξ_C ebenfalls

bekannt ist, und zwar mögen sie sich dort k-fach berühren. Verf. bestimmt die kanonische Schar auf der restlichen Schnittsystem D der beiden Flächen zu

$$\xi_D \equiv (ST-2kCS+T)+(STX) \\ -k(k+1)(CX)+k^2\xi_C-k(k+1)\mu,$$

wo $\mu \equiv (CS)-\sigma \equiv CT-\tau$, σ und τ die Mengen der Punkt-Singularitäten von S und T, X die kanonische Fläche von V_3 bedeuten. *O.-H. Keller* (Dresden).

Tamagawa, Tsuneo. On the theorem of Riemann-Roch. J. Fac. Sci. Univ. Tokyo. Sect. I. 6, 133–144 (1951).

The paper presents a proof of the Riemann-Roch theorem for algebraic functions of one variable which is almost identical to the one given by A. Weil [J. Reine Angew. Math. 179, 129–133 (1938)]. The theorem is then extended to vector spaces over a field of algebraic functions and to semisimple algebras over such a field, giving a new proof of the extension by Witt [Math. Ann. 110, 12–28 (1934)] of the Riemann-Roch theorem to hypercomplex systems. *C. Chevalley.*

Kawahara, Yûsaku. A note on the differential forms on everywhere normal varieties. Nagoya Math. J. 2, 93–94 (1951).

The author shows by an example that a differential form on a normal surface in 3-space may be finite at every simple point of the surface without being of the first kind.

C. Chevalley (New York, N. Y.).

Du Val, Patrick. On regular surfaces of genus three. Canadian J. Math. 3, 148–154 (1951).

En se rapportant à la classification des surfaces algébriques régulières ($p_g=p_a=p$) donnée par F. Enriques dans son livre récent [Le superficie algebriche, Zanichelli, Bologna, 1949; ces Rev. 11, 202], l'auteur s'occupe ici du cas $p=3$. En premier lieu, il démontre en général le théorème suivant (qu'il avait déjà prouvé pour $n=3$): dans un espace projectif à $n+3$ dimensions, la surface F^{4n} commune au cône $V_n{}^4$, qui est la projection d'une surface de Veronese faite d'un espace Ω à $n-3$ dimensions, et à une variété $U_3{}^n$ ($n<10$), dont les sections par des espaces à n dimensions sont des surfaces de Del Pezzo, est le modèle bicanonique d'une surface régulière qui a $p=3$, $p^{(1)}=n+1$. On peut aussi représenter cette surface sur un plan n-ple, dont la courbe de diramation est d'ordre $4n$ et possède $24(n-2)$ points de rebroussement et $8(n-2)(n-3)$ points doubles à tangentes distinctes. Le théorème inverse, qui est déjà connu pour $n=3$ est ici démontré pour $n=4, 5$; dans le sens que, si $n=4, 5$, le modèle bicanonique de la surface régulière la plus générale avec $p=3$, $p^{(1)}=n+1$ peut être obtenue par la construction hyperspatiale indiquée plus haut. On trouve aussi que dans le cas $n=4$, la courbe de diramation du plan quadruple qui représente la surface en question est l'enveloppe d'un système ∞^1 d'index 3 de courbes planes du 4^e ordre, parmi lesquelles il y a deux coniques doubles. L'extension du théorème inverse pour $n \geqq 6$ présente des difficultés, car il semble que la construction donnée ici pour F^{4n} conduit à une surface qui n'est pas la plus générale avec ces valeurs-là des genres. *E. Togliatti* (Gênes).

Godeaux, Lucien. Sur la structure de certains points de diramation de surfaces multiples. Bull. Soc. Roy. Sci. Liège 19, 369–378 (1950).

Une surface algébrique F est dotée d'une involution cyclique d'ordre premier $p=10\nu+1$, n'ayant qu'un nombre fini de points unis. L'image de l'involution est une surface Φ multiple d'ordre p dont les points de diramation A' correspondent aux points unis de F. L'auteur détermine la singularité d'un tel point lorsque le point uni est de seconde espèce, l'homographie correspondante dans le plan tangent étant

$$x_0':x_1':x_2' = x_0:\epsilon x_1:\epsilon^\alpha x_2 \qquad (\epsilon^p=1)$$

avec $\alpha=5\nu+3$. Dans ce cas, A' est multiple d'ordre $\nu+2$; son cône tangent est formé de trois plans et d'un cône rationnel d'ordre $\nu-1$. *B. d'Orgeval* (Grenoble).

d'Orgeval, B. Sur la dégénérescence des surfaces algébriques en systèmes de plans et la dégénérescence des courbes de diramation des plans multiples. Bull. Soc. Roy. Sci. Liège 19, 351–355 (1950).

By considering the multiple plane which is its projection from a general line, it is shewn that if an algebraic surface in four dimensions assumes as the limit of continuous variation a form consisting of a set of planes, then of the complete lines of intersection of these planes by pairs which give rise to double lines in the corresponding limit of the branch curve of the multiple plane, there must be at least three which are concurrent, unless the variable surface is a rational or elliptic ruled surface. *P. Du Val* (Athens, Ga.).

Togliatti, Eugenio. Sulle superficie algebriche col massimo numero di punti doppi. Univ. e Politecnico Torino. Rend. Sem. Mat. 9, 47–59 (1950).

Since the general formula $\frac{1}{2}(n-1)(n-2)$ for the maximum number of nodes of an algebraic plane curve of order n is so easily derived, it long has been assumed that a formula can be obtained defining the maximum number of nodes of an algebraic surface F_n of order n. The author examines critically the past efforts of mathematicians to derive such a formula [J. V. Poncelet, J. Reine Angew. Math. 4, 1–71 (1829); A. B. Basset, Nature 73, 246 (1906); S. Lefschetz, Trans. Amer. Math. Soc. 14, 23–41 (1913); T. R. Hollcroft, J. Reine Angew. Math. 159, 255–264 (1928); F. Severi, Ann. Mat. Pura Appl. (4) 25, 1–41 (1946); these Rev. 9, 609; B. Segre, Boll. Un. Mat. Ital. (3) 2, 204–212 (1947); these Rev. 9, 608]. He shows that all general formulas so far derived for the maximum number of nodes of F_n are incorrect and that this formula cannot be a cubic polynomial in n. He states that the problem is associated with both Plücker's equations and the properties of the "still little known" moduli of an algebraic surface, and concludes that it is not only not now possible to predict a definite result but that probably it will not be until further research has shown the existence of surfaces of orders $n \geqq 5$ "with many nodes".

T. R. Hollcroft (Aurora, N. Y.).

Semple, J. G. Note on Halphen conditions. J. London Math. Soc. 26, 122–125 (1951).

In a plane π there are ∞^3 Halphen conics, i.e., ∞^3 complete conics each of which consists of a repeated line as locus and a repeated point of this line as envelope. The author calls Halphen condition any simple condition, on the complete conics of π, which is satisfied by all Halphen conics. On the minimum nonsingular model $\Omega_5{}^{102}$[27] of all the complete conics k of π, the Halphen conics are represented by a nonsingular threefold θ, of order 48. Any fourfold cut on Ω by a quadric through θ represents a Halphen condition ρ, a simple concrete example of which is given by the condition on a conic k of π to have some 4-point contact with some conic of a given polar net of conics. This condition ρ, together with the two usual fundamental conditions μ and ν, forms

a base covering Halphen conditions of the simplest type; and all the numbers $\mu^\alpha\nu^\beta\rho^\gamma$ ($\alpha+\beta+\gamma=5$; $\alpha\geqq 0$, $\beta\geqq 0$, $\gamma\geqq 1$) are obtained, e.g., $\rho^5=1296$. Finally, the effect of a birational transformation of Ω resolving θ into a fourfold is investigated.

B. Segre (Rome).

Hall, R. On the representation of rational sections of the Grassmannian of lines of five dimensions. Proc. Cambridge Philos. Soc. 47, 305–308 (1951).

The two starting points of this paper are as follows. First, the Grassmannian V_8^{14}[14] of lines of S_5 can be mapped birationally on an S_8 in such a way that its prime sections correspond to quadrics through a V_4^4[7] generated by ∞^1 solids. Secondly, if q quadrics of S_r contain a common $(q-1)$-dimensional space Π and are such that a general $[q]$ through Π meets them residually in $[q-1]$'s which meet in a point P not lying in Π, then the locus of P—the whole or an irreducible component of the intersection of the quadrics —is necessarily a rational V_{r-q}. By applying the second result to the first representation, with a point, line or plane in a solid of V_4^4 as Π, the author derives birational representations of sections of V_8^{14} by general spaces S_{13}, S_{12}, S_{11} or spaces S_7, S_6, S_5 respectively. These representations are of orders 3, 4 and 5. Partial descriptions of the bases in S_7, S_6 and S_5 are given, excluding, however, the base varieties arising from V_4^4. The third result, as stated, is incorrect, the error originating on line 14 of p. 307, where the symbol ψ_3^8 should have been ψ_3^7; also it is not made clear why the author's method cannot be applied to the section of V_8^{14} by a general S_{10}. The last section deals with extensions of the results to rational sections of the Grassmannian of lines of S_n.

J. G. Semple (London).

Differential Geometry

Bouligand, Georges. Sur les transformations de contact réelles. C. R. Acad. Sci. Paris 232, 911–913 (1951).

$\mathbf{OM}=\mathfrak{X}(x_1,\cdots,x_n,z,p_1,\cdots,p_n)$ représente un vecteur de l'espace euclidien E_{n+1} fonction de l'élément de contact du premier ordre $(x_1,\cdots,x_n,z,p_1,\cdots,p_n)$ admettant des dérivées premières continues. Si le point $\mathbf{m}(x_1,\cdots,x_n,z)$ décrit une n-surface $z=\varphi(x_1,\cdots,x_n)$ où φ a des dérivées secondes continues, $\mathbf{M}$ décrit une image dont le n-plan tangent est, sauf exception, déterminé par les n-vecteurs $\mathbf{V}_i=\mathbf{M}_{x_i}+p_i\mathbf{M}_z+p_{i1}\mathbf{M}_{p_1}+\cdots+p_{in}\mathbf{M}_{p_n}$, $i=1,2,\cdots,n$. Si $\mathfrak{X}$ définit une transformation de contact, le rang du système des vecteurs $\mathbf{M}_{x_i}+p_i\mathbf{M}_z$, $\mathbf{M}_{p_j}$ qui ne dépend que de $(\mathbf{m},p_i)$ est en général $=n$. L'élément de contact du second ordre $(\mathbf{m},p_h,p_{ij})$ est dit ordinaire si les $\mathbf{V}_i$ sont linéairement indépendants, critique s'ils sont linéairement dépendants. Le rôle des éléments critiques est explicité dans le cas où $\mathbf{OM}=\mathbf{Om}+l\mathbf{r}(\mathbf{m},p_i)$, supposant qu'il s'agisse d'une transformation de contact quelle que soit la constante l. Des indications sont données concernant les n-surfaces ne portant que des éléments critiques dans le cas où la transformation de contact est définie par $k+1$ équations directrices. Pour $n=2$, $k=0$, l'auteur définit des systèmes conjugués généralisés sur la surface $\mathbf{m}=\mathbf{m}(u,v)$ en demandant que la limite de l'intersection des antécédentes de $\mathbf{M}$ et $\mathbf{M}+\mathbf{M}_u du$ soit tangente à $\mathbf{m}_v$ et signale que la dualité des systèmes conjugués usuels reste valide dans cette extension.

C. Pauc (Le Cap).

Moór, Arthur. Erweiterung des Vierscheitelsatzes auf dreidimensionale Kurven. Duke Math. J. 18, 509–516 (1951).

The proof of the four-vertex theorem for plane ovals which was given by Blaschke [Vorlesungen über Differentialgeometrie . . . , Bd. 1, Springer, Berlin, 1930, pp. 30–32] and credited by him to G. Herglotz depends on the vanishing of the integral $\oint_C \kappa\xi_i ds$ where κ denotes the curvature of the oval and ξ_i the unit tangent vector. Let a closed space curve C have the property that through every two points of C there can be passed a plane cutting C nowhere else. For such a curve C it follows in exactly the same way that a function $\varphi(s)$ defined on C has at least four extrema if $\oint_C \varphi\xi_i ds=0$. The author finds a class of such functions φ which are invariants of C, the class depending on an arbitrary function. These invariants are interpreted geometrically in certain special cases. Since the functions φ obtained involve, in general, the reciprocal of the torsion, it seems to be tacitly, though not explicitly, assumed that the torsion is nonvanishing. This tacit assumption renders theorem 4 of the paper vacuous, since it deals with closed spherical curves and such curves always have zeros of the torsion. If this assumption is not made, the proof of theorem 4 is invalid.

S. B. Jackson (College Park, Md.).

Strubecker, Karl. Über monofokale Kegelschnitte. Math. Nachr. 4, 36–46 (1951).

A theorem, due to O. Emersleben and T. Pöschl, concerning a system of ellipses with a fixed focus S', fixed length $2a$ of the major axis and passing through a fixed point P' is proved and extended by means of the methods of descriptive geometry [cf. O. Emersleben, Math. Nachr. 3, 62–70 (1949); these Rev. 11, 534; T. Pöschl, Einführung in die analytische Mechanik, Braun, Karlsruhe 1949, pp. 24–26].

W. van der Kulk (Providence, R. I.).

Wunderlich, Walter. Die Haupttangentenkurven gewisser metrisch spezieller Flächen 3. Ordnung. Anz. Öster. Akad. Wiss. Math.-Nat. Kl. 1950, 143–147 (1950).

Hohenberg [Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. 1949, 287–290; diese Rev. 11, 682] hat eine elementare Ermittlung der Haupttangentenkurven einer metrisch speziellen Fläche 3. Ordnung mit einem biplanaren und zwei konischen Knotenpunkten mitgeteilt. Verf. weist auf ein anderes Prinzip zur Bestimmung der Kurven hin, das unmittelbar auf der Erzeugung der genannten Fläche aus einer Ebene vermöge einer axialen Inversion beruht. Diese Methode führt zu einer Erweiterung, da sie anwendbar ist auf diejenigen Flächen welche man mittels einer "verallgemeinerten axialen Inversion" aus einer Ebene erhalten kann; es handelt sich dann um metrisch spezielle Flächen 3. Ordnung mit vier konischen Knotenpunkten. Die Haupttangentenkurven erweisen sich als rationale Kurven 6. Ordnung.

O. Bottema (Delft).

Wunderlich, Walter. Eine kennzeichnende Eigenschaft der D-Linien von Quadriken. Monatsh. Math. 55, 76–81 (1951).

The D-curves of a surface F are those curves on F whose osculating spheres are tangent spheres of F. Three theorems are proved on the D-curves of a quadric Φ. Let c denote a conic through the four points in which Φ meets the absolute circle. Then theorem 1 states that a curve on Φ is a D-curve if and only if its tangent surface intersects the infinite plane along a conic c. Theorem 2: "Bei jeder allgemeinen D-Linie einer Quadrik hängen die Punkte mit den zugehörigen

Schmiegkugelmitten durch eine Raumaffinität zusammen, welche die Hauptsymmetrieebenen der Fläche festlässt." Theorem 3: "Besteht bei einer reellen Raumkurve ein affiner Zusammenhang zwischen ihren Punkten und Schmiegkugelmitten, dann ist sie D-Kurve einer Quadrik."

P. Scherk (Saskatoon, Sask.).

Hauer, F. Die flächentreue Meridianstreifenabbildung des Rotationsellipsoids in die Ebene im Vergleiche mit der flächentreuen querachsigen Zylinderabwicklung. Österreich. Z. Vermessgswes. **39**, 10–17 (1951).

In the immediate neighborhood of a reference meridian the subject projections are shown to be equivalent to third powers of the longitudinal displacement. Similar consideration is also given to a transverse perspective cylinder development of the sphere. *N. A. Hall.*

Mishra, R. S. On the congruences of Ribaucour. Ganita **1**, 5–9 (1950).

L'auteur donne, pour les coefficients D, D', D'' de la deuxième forme fondamentale de la surface de référence d'une congruence rectiligne de Ribaucour, des expressions particulières susceptibles de diverses applications.

P. Vincensini (Marseille).

Mishra, R. S. On rectilinear congruences. Ganita **1**, 45–52 (1950).

Si, pour une congruence rectiligne rapportée à une surface de référence arbitraire, t_1 et t_2 sont les abscisses des deux points limites situés sur un rayon quelconque et d_1, d_2 les paramètres principaux relatifs à ce même rayon, l'auteur établit la relation $t_1-t_2=d_1-d_2$. Il rattache à cette relation un certain nombre de propriétés des congruences rectilignes les plus générales ou bien des congruences normales ou de Ribaucour. *P. Vincensini* (Marseille).

Süray, Saffet. Sur les surfaces réglées d'une congruence rectiligne. Communications Fac. Sci. Univ. Ankara **2**, 11–47 (1949).

En partant du point de vue de Sannia, l'auteur se propose de faire l'étude des propriétés différentielles métriques des surfaces réglées d'une congruence rectiligne qui passent par une génératrice donnée. Parmi ces surfaces il a envisagé celles dont les normales à leurs points centraux sont parallèles à deux diamètres conjugués de l'indicatrice des paramètres de distribution (la notion analogue à l'indicatrice des courbures d'une surface: Si

$$r=r_1(u,v)+\zeta r_2(u,v) \quad \text{où} \quad |r_1|^2=1$$

est l'équation vectorielle de la congruence et a est la droite de la congruence et π le plan normal en point A de la droite a, le lieu géométrique dans le plan π due point Q défini par

$$\overrightarrow{AQ}=\sqrt{p}\frac{dr_2}{\sqrt{|dr_2|^2}},$$

où p est le paramètre de distribution de la surface réglée qui a la normale centrale $dr_2/\sqrt{|dr_2|^2}$, est l'indicatrice des paramètres de distribution relative à la génératrice a), celles dont les normales aux points centraux sont orthogonales, etc. Aussi les cas de quelques congruences spéciales, par ex. d'une congruence de normales, sont étudiés.

F. Vyčichlo (Prague).

Hsiung, Chuan-Chih. Conjugate nets in three- and four-dimensional spaces. Duke Math. J. **18**, 487–499 (1951).

Let Φ be a fixed plane and P_x be a nonsingular point of a conjugate net N_x in ordinary space. The points M, M' of intersection of the fixed plane Φ and the two tangents of the net N_x at the point P_x describe two plane nets N_M, $N_{M'}$ respectively. The author proves that one of the two plane nets N_M, $N_{M'}$ is a Laplace transformed net of the other. This result is also extended to a linear space S_4 by using a fixed hyperplane instead of the fixed plane Φ. Let Z_1, Z_2 be the points of intersection of the asymptotic tangents to a surface S at a nonsingular point x with the fixed plane Φ. The points Z_1, Z_2 describe plane nets N_{Z_1}, N_{Z_2} which have equal and nonzero Laplace-Darboux invariants if, and only if, the surface S is isothermally asymptotic. Let Ψ be a fixed plane determined by two fixed hyperplanes in the space S_4, and N_T be the plane net described by the point T of intersection of the fixed plane Ψ with the tangent plane at a point x of the net N_x. The equation of Laplace and the Laplace-Darboux invariants of the net N_T are derived, and some special cases are studied in which one or both of the first and minus-first Laplace transformed nets of the net N_T degenerate into curves or the net N_T has equal and nonzero Laplace-Darboux invariants. *P. O. Bell.*

Lemoine, Simone. Sur les réseaux conjugués persistants à angle constant. C. R. Acad. Sci. Paris **232**, 1630–1631 (1951).

It is proved that the cylindrical moulding surfaces are the only surfaces which can be deformed in such a way that there is an invariant conjugate net whose curves cut each other at a constant angle. *S. Chern* (Chicago, Ill.).

Jeger, M. Topologische Fragen der Differentialgeometrie. Projektive Methoden in der Gewebegeometrie. Comment. Math. Helv. **24**, 260–290 (1950).

Diese Arbeit stimmt im wesentlichen mit der Promotionsarbeit "Projektive Zusammenhänge und Gewebe" des Verf. überein [Thesis, Eidgenössische Technische Hochschule in Zürich, 1949; diese Rev. **11**, 211]. *O. Varga.*

Bell, P. O. A theorem of Cartan. Duke Math. J. **17**, 453–455 (1950).

Consider a field of contact elements (M, P) in a real projective space of $r\geqq 3$ dimensions. To each point M is thus attached a distinct hyperplane P according to a determined law. The field is said to be a field (C) if every curve which is tangent at each of its points M to the hyperplane P has at M contact of the second order with P. The fields (C) which are extensible to all of the space are those for which the hyperplane P associated with a point M is the focal hyperplane of M with respect to a fixed linear complex [É. Cartan, Ann. Sci. École Norm. Sup. (3) **62**, 205–231 (1945); these Rev. **8**, 92]. This note contains an elementary proof of this theorem. *O. Borůvka* (Brno).

Poznyak, È. G. Infinitesimal deformations of polygonal troughs. Doklady Akad. Nauk SSSR (N.S.) **78**, 205–207 (1951). (Russian)

Let Π', Π'' be two parallel planes in 3 dimensions and for $i=1, 2, \cdots, n$, let S_i be a piece of ruled surface, with generators parallel to these planes, which is bounded by two generators and two curved edges L_i, L_{i+1}. By gluing the S_i along the curved edges in circular order, the author obtains a surface whose boundary consists of two plane polygons. He terms this surface a polygonal trough. In the case $n=3$ he finds that the trough is rigid under infinitesimal deformations whose field of translation is single-valued on the edges. For general n, he studies rigidity under infinitesimal deformations for which the field of rotation is also single-

valued on the edges. He finds that rigidity in this sense is equivalent to the existence of at least two sections of the trough by planes parallel to Π', Π'' which define nonsimilar polygons of nonzero area. Proofs are omitted.

L. C. Young (Madison, Wis.).

Lyra, Gerhard. Über eine Modifikation der Riemannschen Geometrie. Math. Z. 54, 52–64 (1951).

In an n-dimensional manifold referred to a coordinate system (x^i), the author defines the components of a vector whose origin and end-point are respectively $P(x^i)$ and $P'(x^i+dx^i)$ as $\xi^i=x^0dx^i$, where the factor x^0 is a continuous function, different from zero, of x^i. The function x^0 is called a gauge. The components of the vector ξ^i are determined when the coordinate system (x^i) and the gauge system (x^0) are fixed, or briefly, when the reference system $(x^i; x^0)$ is fixed. The most general transformation of the reference system must be of the form

$$\begin{cases} \bar{x}^i=\bar{x}^i(x^1, x^2, \cdots, x^n), \\ \bar{x}^0=\bar{x}^0(x^1, x^2, \cdots, x^n; x^0(x^1, x^2, \cdots, x^n)) \end{cases}$$

where $|\partial\bar{x}^i/\partial x^a|\neq 0$ and $\partial\bar{x}^0/\partial x^0\neq 0$.

Under this transformation of reference system, the transformation law of the components of a contravariant vector ξ^i and that of the components of a covariant vector ξ_i are, respectively,

$$\bar{\xi}^i=\lambda\frac{\partial\bar{x}^i}{\partial x^a}\xi^a \quad\text{and}\quad \bar{\xi}_i=\lambda^{-1}\frac{\partial x^a}{\partial\bar{x}^i}\xi_a \qquad (\lambda=\bar{x}^0/x^0),$$

the transformation law of the components of a general tensor being evident from these examples. The covariant derivative $T^{\cdots}_{\cdots;k}$ of a tensor $T^{\cdots}_{\cdots}$ having r contravariant and s covariant indices is defined to be

$$T^{\cdots}_{\cdots;k}=\frac{1}{x^0}\frac{\partial T^{\cdots}_{\cdots}}{\partial x^k}+\sum T^{\cdot a\cdot}_{\cdots}\Gamma^i_{ak}-\sum T^{\cdots}_{\cdot a\cdot}\Gamma^a_{jk}-\frac{r-s}{2}T^{\cdots}_{\cdots}\varphi_k.$$

Under a coordinate transformation, the functions Γ^i_{jk} and φ_k, introduced here, have respectively the ordinary transformation law of an affine connection and that of a covariant vector. Under a gauge transformation, they have the following transformation laws:

$$\bar{\Gamma}^i_{jk}=\lambda^{-1}\Gamma^i_{jk}, \quad \bar{\varphi}_k=\frac{1}{\lambda}\left(\varphi_k+\frac{2}{x^0}\frac{\partial\log\lambda}{\partial x^k}\right).$$

The author discusses then the auto-parallel curves and curvatures. To treat the metric case, the author defines the square of the length of a vector $\xi^i=x^0dx^i$ by a positive definite quadratic form $g_{jk}\xi^j\xi^k$, the coefficients g_{jk} being components of a covariant tensor. If we assume that the length of a vector is invariant under the parallel displacement, then we have

$$\Gamma^i_{jk}=\frac{1}{x^0}\left\{ {i \atop jk} \right\}+\tfrac{1}{2}(\delta_j^i\varphi_k+\delta_k^i\varphi_j-g^{ia}\varphi_a g_{jk}).$$

In the last section of the paper, the author gives a comparative study of Weyl geometry and his own geometry. In Weyl geometry, it is assumed that the covariant differential of the square of the length of a vector is proportional to itself when the vector is displaced by parallelism, while, in the present geometry, it is assumed that it is zero.

K. Yano (Princeton, N. J.).

Lichnerowicz, André. Sur les formes harmoniques des variétés riemanniennes localement réductibles. C. R. Acad. Sci. Paris 232, 1634–1636 (1951).

If a compact orientable Riemannian manifold V_n carries a k-vector field whose covariant derivative is zero, the metric on V_n is locally reducible, that is, it can be written locally as

$$ds^2=\sum_{i,j=1}^{k} g_{ij}(x^1, \cdots, x^k)dx^idx^j+\sum_{\alpha,\beta=k+1}^{n} g_{\alpha\beta}(x^{k+1}, \cdots, x^n)dx^\alpha dx^\beta$$

[the author, same vol., 146–147, 677–679 (1951); these Rev. 12, 536, 746]. A p-form can be expressed uniquely as a sum of pure forms, a pure form being homogeneous in $dx^1, \cdots, dx^k$. It is shown that if the metric on V_n is locally reducible, the pure components of any harmonic form are harmonic. In the case in which the constant k-vector field defines a fibring on V_n, it is further shown that any pure harmonic p-form can be written as a finite sum $\sum\phi_i\psi_i$, where ϕ_i depends only on $x^1, \cdots, x^k, dx^1, \cdots, dx^k$ and is harmonic, and ψ_i depends only on $x^{k+1}, \cdots, x^n, dx^{k+1}, \cdots, dx^n$ and is harmonic.

W. V. D. Hodge (Cambridge, England).

Patterson, E. M. An existence theorem on simply harmonic spaces. J. London Math. Soc. 26, 238–240 (1951).

On sait qu'un espace riemannien R_n est simplement harmonique si, pour tout choix d'un point fixe Q, l'équation de Laplace associée à R_n admet pour solution élémentaire S^{2-n}, où S est la distance géodésique au point Q. Jusqu'ici les seuls exemples connus d'espaces simplement harmoniques étaient soit symétriques au sens de Cartan, soit récurrents au sens de Ruse [Proc. London Math. Soc. (2) 50, 317–329 (1948); ces Rev. 10, 266]. Le principal théorème de ce papier est le suivant: tout espace riemannien R_{2p} admettant p champs parallèles indépendants de vecteurs nuls, mutuellement orthogonaux, est simplement harmonique. Pour caractériser les espaces simplement harmoniques, l'auteur utilise une méthode dont le principe est du à A. G. Walker [Proc. Edinburgh Math. Soc. (2) 7, 16–26 (1942); ces Rev. 4, 171]. De ce théorème, l'auteur déduit que, pour toutes les dimensions paires, il existe des espaces simplement harmoniques qui ne sont ni symétriques ni récurrents.

A. Lichnerowicz (Paris).

Buchdahl, H. A. On the Hamiltonian derivatives arising from a class of gauge-invariant action principles in a W_n. J. London Math. Soc. 26, 139–149 (1951).

Soit W_n un espace de Weyl à n dimensions, $g_{\mu\nu}$ et k_ν les coefficients de ses formes fondamentales, quadratique et linéaire, $K|g|^{\frac{1}{2}}$ une densité tensorielle de poids $+1$ admettant l'invariance de jauge. Les équations du champ unitaire dérivant du principe variationnel déduit de $K|g|^{\frac{1}{2}}$ fait intervenir les dérivées hamiltoniennes de K par rapport aux $g_{\mu\nu}$ et k_μ. L'auteur se propose d'obtenir des formules commodes pour ces dérivées $P^{\mu\nu}$ et Q^μ, dans l'hypothèse où K fait intervenir seulement les $g_{\mu\nu}$, leurs dérivées jusqu'au second ordre (notées $g_{\mu\nu,\sigma\rho}$), les k_μ et leurs dérivées du premier ordre $k_{\mu,\nu}$; les formules établies sont les suivants

$$P^{\mu\nu}=\nabla_\rho\nabla_\sigma Z^{\mu\nu\sigma\rho}-\tfrac{2}{3}Z^{\sigma\rho\alpha(\mu}B_{\sigma\rho\alpha}{}^{\nu)}+\tfrac{1}{2}g^{\mu\nu}K-\tfrac{1}{2}S^{\sigma(\mu}F_\sigma{}^{\nu)},$$

$Q^\mu=-\nabla_\nu S^{\mu\nu}$ où $B_{\mu\nu\sigma}{}^\rho$ est le tenseur de courbure de Weyl admettant l'invariance de jauge où $Z^{\mu\nu\sigma\rho}=\partial K/\partial g_{\mu\nu,\sigma\rho}$, $S^{\mu\nu}=\partial K/\partial k_{\mu,\nu}$, $E_{\mu\nu}=2k_{[\mu,\nu]}$, et où les autres notations sont celles de Schouten. La méthode de démonstration utilise fréquemment des considérations d'invariance pour éviter de longs calculs. Le tenseur $E^{\mu\nu\sigma\rho}=\partial K/\partial B_{\mu\nu\sigma\rho}$ y joue un rôle important. Le papier comporte en outre une étude du cas où le principe variationnel est fondé sur une densité scalaire $\mathfrak{K}$ faisant intervenir les densités tensorielles numériques $\epsilon^{\mu_1\cdots\mu_n}$ et $\epsilon_{\mu_1\cdots\mu_n}$ de Levi-Civita; il se termine par un exemple relatif au cas $n=4$.

A. Lichnerowicz (Paris).

Buchdahl, H. A. An identity between the Hamiltonian derivatives of certain fundamental invariants in a W_4. J. London Math. Soc. **26**, 150–152 (1951).

Les notations de ce papier étant celles de l'analyse ci-dessus, l'auteur envisage dans un W_4 les invariants K qui sont des polynomes par rapport aux composantes du tenseur de courbure. On sait que ce sont des combinaisons linéaires à coefficients constants des quatre invariants quadratiques: $K_1=G^2$, $K_2=G_{\mu\nu}G^{\mu\nu}$, $K_3=B_{\mu\nu\sigma\rho}B^{\mu\nu\sigma\rho}$, $K_4=F_{\mu\nu}F^{\mu\nu}=\frac{1}{4}G_{\mu\nu}G^{[\mu\nu]}$, où $G_{\mu\nu}$ désigne le tenseur contracté de courbure. Généralisant un résultat de Lanczos [Ann. of Math. (2) **39**, 842–850 (1938)] l'auteur montre que l'invariant

$$L=K_1-4K_2+K_3+12K_4$$

admet des dérivées hamiltoniennes identiquement nulles. La méthode fait intervenir d'une manière essentielle les résultats du papier précédent. Il en résulte que, pour former un système d'équations du champ unitaire sur la base d'un invariant tel que K, on peut ne considérer que les invariants K_1, K_2, et K_4, c'est-à-dire ceux qui ne font intervenir que les composantes du tenseur contracté de courbure.

A. Lichnerowicz (Paris).

Tashiro, Yoshihiro. Sur la dérivée de Lie de l'être géométrique et son groupe d'invariance. Tôhoku Math. J. (2) **2**, 166–181 (1950).

Verf. definiert zunächst für einen n-dimensionalen Raum X_n das geometrische Objekt. Seine Definition entspricht der ursprünglichen von O. Veblen und J. M. Thomas, ist aber insofern allgemeiner, da in dem Transformationsgesetz auch die ursprünglichen und neuen Koordinaten der X_n auftreten können. Das übliche geometrische Objekt wird hier als differenzielles geometrisches Objekt bezeichnet. Für das so eingeführte geometrische Objekt wird das zu einer infinitesimalen Transformation gehörige Liesche Differenzial bzw. die Liesche Ableitung auf bekannte Weise definiert und die entsprechenden analytischen Ausdrücke angegeben. Aus ihnen ergibt sich unmittelbar, dass die Liesche Ableitung dann und nur dann wieder ein geometrisches Objekt darstellt, falls die Transformationsformeln des vorgegebenen Objektes in den Komponenten desselben linear sind. Solche Objekte werden vom Verf. als linear bezeichnet. Bei Zugrundelegung einer kontinuierlichen Transformationsgruppe wird der Klammerausdruck einer skalaren Funktion auf lineare geometrische Objekte ausgedehnt. Die Invarianzgruppe eines Objektes wird von denjenigen infinitesemalen Transformationen erzeugt, die die Lieschen Ableitungen desselben zum verschwinden bringen. Über solche Gruppen werden einige Sätze gebracht. Schliesslich erklärt Verf. auch Objekte höherer Ordnung, die dadurch bestimmt sind, dass in ihrem Transformationsgesetz Differenziale bzw. (bei Einführung von Untermannigfaltigkeiten des) partielle Ableitungen höherer Ordnung auftreten. Die Lieschen Ableitungen dieser Objekte stehen mit den erweiterten Gruppen in Zusammenhang.

O. Varga (Debrecen).

Moreau, Jean-Jacques. La symétrie de révolution en calcul tensoriel. C. R. Acad. Sci. Paris **231**, 1028–1029 (1950).

The author constructs the expressions for the purely isotropic tensors of the plane and also for those tensors of three space which possess symmetry of revolution (but not reflection) with respect to a specified direction. The tensors of the first four orders are determined. In the theory of isotropic homogeneous turbulence, the corresponding tensors of the first three orders which permit reflections are well known [von Kármán and Howarth, Proc. Royal Soc. London. Ser. A. **164**, 192–215 (1938)].

N. Coburn.

NUMERICAL AND GRAPHICAL METHODS

★Tables Relating to Mathieu Functions. Characteristic Values, Coefficients, and Joining Factors. Prepared by The Computation Laboratory of the National Applied Mathematics Laboratories, National Bureau of Standards. Columbia University Press, New York, N. Y., 1951. xlviii+278 pp. $8.00.

Es werden die zwei Hauptformen der Mathieuschen Differentialgleichung an die Spitze der Überlegungen gestellt. Es ist sehr zu hoffen, dass die hier benutzte Schreibweise nun bei den Autoren aller Länder Eingang finden wird. Die geraden und ungeraden periodischen Lösungen mit periodischen Koeffizienten werden angeschrieben und normiert. Ebenso werden die nicht periodischen Lösungen der zweiten Mathieuschen Differentialgleichung (mit hyperbolischem Koeffizienten) angeschrieben und normiert. Die entsprechenden Eigenwerte werden angegeben und ihre Berechnung wird diskutiert. Die asymptotischen Ausdrücke für die verschiedenen Mathieuschen Funktionen werden angeschrieben. Die verschiedenen auftretenden Koeffizienten werden diskutiert. Schliesslich werden die Berechnungsverfahren angegeben. Die jetzt vorliegenden Tafeln werden sich für alle Rechnungen mit diesen Funktionen als ausserordentlich nützlich und unerlässlich erweisen.

M. J. O. Strutt (Zürich).

Bartberger, C. L. The magnetic field of a plane circular loop. J. Appl. Phys. **21**, 1108–1114 (1950).

"The purpose of this paper is to publish tables of values of (the) two integrals [occurring in the problem referred to in the title] together with formulas for their use." The integrals are

$$I_1=(1/\pi)\int_0^\pi(1-b\cos\theta)^{-\frac{1}{2}}d\theta,$$

$$I_2=(1/\pi)\int_0^\pi(1-b\cos\theta)^{-1}\cos\theta d\theta,$$

which are listed for $b=0(0.001)1$ [6D]. In addition, there are listed I_1-I_2, $(1-b)I_1$, $(1-b)\,I_2$ for $b=0.8(0.001)1$ [6D], and I_1, I_2, I_1-I_2, $(1-b)I_1$, $(1-b)I_2$ for $b=0.995(0.0001)1$ [7 or 8S].

C. J. Bouwkamp (Eindhoven).

Borel, Émile. Les décimales de e et de π. C. R. Acad. Sci. Paris **232**, 1973–1974 (1951).

J. von Neumann has observed the abnormally small variance of the frequencies of the digits in the first 2500 digits of e, computed by the machine. It is pointed out here that this actually corresponds to the minimum value of the variance so far. Perhaps the conditional probability, knowing this, rather than the a priori one, should be taken into consideration. Three blocks (the fourth being an obvious mistake) of 10 consecutive distinct digits are also observed.

K. L. Chung (Ithaca, N. Y.).

Gill, S. The diagnosis of mistakes in programmes on the EDSAC. Proc. Roy. Soc. London. Ser. A. **206**, 538–554 (1951).

In this paper the author gives an exposition of the various techniques used to check for mistakes in coding or program-

ming tapes used in connection with the EDSAC, the computer at the Mathematical Laboratory, University of Cambridge. He describes the principles followed and illustrates with a detailed account of one of their checking routines. *H. H. Goldstine* (Princeton, N. J.).

Booth, Andrew D. A signed binary multiplication technique. Quart. J. Mech. Appl. Math. **4**, 236–240 (1951).

Haldane, J. B. S. The extraction of square roots. Math. Gaz. **35**, 89–90 (1951).

Loud, W. S. The probability of a correct result with a certain rounding-off procedure. Proc. Amer. Math. Soc. **2**, 440–446 (1951).

By use of characteristic functions, the following theorem is proved: For $i=1, 2, \cdots, n$, let x_i be uniformly distributed real random variables expressed in the base B of numeration, and let $\bar{x}_i$ be the nearest integer to x_i. Let S and $\bar{S}$ be the nearest multiples of B to $\sum_1^n x_j$ and $\sum_1^n \bar{x}_j$, respectively. Then the probability that $S=\bar{S}$ is $(2/\pi B)\int_0^\infty \phi(t)dt$, if B is odd, and $(2/\pi B)\int_0^\infty \phi(t)\cos t dt$, if B is even, where $\phi(t)=t^{-2}(t^{-1}\sin t)^{n-1}\sin^2 Bt$. The result is independent of the location of the B-ary point. Carrying k more places in the $\bar{x}_j$ than in $\bar{S}$ is treated with the theorem, using the base B^k. Asymptotic probabilities are given as $n\to\infty$ and as $B\to\infty$. If $n=2000$ and $B=10$, the probability that $S=\bar{S}$ is .99 if $k=3$ and .999 if $k=4$. *G. E. Forsythe.*

Tyuring, A. [Turing]. Rounding off errors in matrix processes. Uspehi Matem. Nauk (N.S.) **6**, no. 1(41), 138–162 (1951). (Russian)

Translated from Quart. J. Mech. Appl. Math. **1**, 287–308 (1948); these Rev. **10**, 405.

Jaeckel, K. Fehlerausgleichung bei Funktionen in Parameterdarstellung. Z. Angew. Math. Mech. **31**, 185–186 (1951).

Salzer, Herbert E. Checking and interpolation of functions tabulated at certain irregular logarithmic intervals. J. Research Nat. Bur. Standards **46**, 74–77 (1951).

A method is given for checking and interpolation of functions which behave like polynomials in $\log x$ and are tabulated at irregular intervals based on some or all of the points 1, 2, 5, 10, 20, 50, etc. *T. N. E. Greville.*

Michalup, Eric. On osculatory interpolation. Estados Unidos de Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. **12**, no. 39, 28–75 (1949). (Spanish)

After a survey of the history of smooth-junction interpolation, criteria to be adopted for judging the smoothness of an interpolation are discussed. *T. N. E. Greville.*

Gebelein, Hans. Anwendung gleitender Durchschnitte zur Herausarbeitung von Trendlinien und Häufigkeitsverteilungen. Mitteilungsblatt Math. Statist. **3**, 45–68 (1 plate) (1951).

The paper deals with the fitting of curves, such as trend lines or frequency functions, to observed data. The author discusses the method of moving averages which in his opinion is preferable to least squares. The paper is purely descriptive, no attempt is made to apply modern methods of statistical inference. *E. Lukacs* (Washington, D. C.).

Kolscher, M. Die Berechnung vollständiger elliptischer Integrale dritter Gattung durch Reihen. Z. Angew. Math. Mech. **31**, 114–120 (1951). (German. English, French, and Russian summaries)

Series are given for evaluating

$$\int_0^{\pi/2} \frac{(1-k^2\sin^2\varphi)^{\frac{1}{2}}}{1-\lambda^2\sin^2\varphi}d\varphi, \quad 0\leqq k^2\leqq 1,\ \lambda^2>0.$$

If $\lambda^2>1$ the Cauchy principal value is understood. The series are given in ascending powers of k^2, convergent when $k^2<1$, and in ascending powers of $k'^2=1-k^2$, convergent when $k'^2<1$. The series are practically useful only when k^2 or k'^2 are sufficiently small. Should this condition not hold, the application of the descending (or ascending) Landen transformation will bring the parameter within the effective range. *L. M. Milne-Thomson* (Greenwich).

Soulé-Nan, Geneviève, et Peltier, Jean. Méthode de calcul des intégrales de la forme

$$J_{CS}=\int_0^\infty \frac{\cos k\sqrt{\lambda^2+\alpha^2}}{\sqrt{\lambda^2+\alpha^2}}\sin p\lambda d\lambda.$$

C. R. Acad. Sci. Paris **232**, 2076–2078 (1951).

The integrals J_{CS} of the title and analogous integrals J_{CC}, J_{CS}, J_{SS} (indices indicate cosine or sine) arise in antenna theory. The authors express the four integrals in terms of the functions

$$S(a, X)=\int_0^X (\sin\gamma/\gamma)dx, \quad C(a, X)=\int_0^X \{(1-\cos\gamma)/\gamma\}dx,$$

where $\gamma=\sqrt{x^2+a^2}$. The functions $S(a, X)$ and $C(a, X)$ are tabulated for values of a corresponding to $p/k<1$, but not to $p/k>1$. The authors give an approximation to J_{CS}, J_{CC}, etc., valid for $1<p/k<1+\epsilon$. *G. E. Forsythe.*

Kopal, Zdeněk, Carrus, Pierre, and Kavanagh, Katherine E. A new formula for repeated mechanical quadratures. J. Math. Physics **30**, 44–48 (1951).

Les auteurs donnent cette formule:

$$\int_{-1}^{+1}\int_{a_1}^{x}\phi(x)dx=\sum_{j=e}^{n} H_j^{(n)}\phi(a_j),$$

exacte pour les polynomes de degré $2n-1$, où les a_j sont des nombres algébriques convenables dépendant de n. Pour n pair a_1 est irrationnel; pour n impair on peut toujours prendre $a_1=0$. Ils donnent valeurs des a_j et $H_j^{(n)}$ pour $n=2, 3, 4, 7$ avec 7 decimals. *J. Kuntzmann.*

Aitken, A. C. Studies in practical mathematics. VI. On the factorization of polynomials by iterative methods. Proc. Roy. Soc. Edinburgh. Sect. A. **63**, 174–191 (1951).

In this paper there is given the theory of an iterative method for the approximation of an exact factor of a polynomial as introduced by S. N. Lin [J. Math. Physics **20**, 231–242 (1941); these Rev. **3**, 153]. The process is extended to divisors of arbitrary degree. It is treated from the point of view of repeated linear transformation of the vector of small errors or deviations from the final exact coefficients. The matrix governing the iterative process is obtained, and its latent roots and latent vectors are found. The convergence of the process is discussed, and processes are developed for the acceleration of convergence. Numerical examples illustrating various cases that arise are given.
E. Frank (Chicago, Ill.).

Gornšteĭn, M. S. The numerical solution of equations. Doklady Akad. Nauk SSSR (N.S.) 78, 193–196 (1951). (Russian)

Let $f(x)$ have a zero x_0 in the interval (a, b). Let ξ in (a, b) and an integer $n \geqq 0$ be fixed. Choose $\phi_n(x;\xi) = x - H_n(x;\xi)f(x)$ so that $\phi_n'(\xi;\xi) = \cdots = \phi_n^{(n+1)}(\xi;\xi) = 0$ (derivatives with respect to x). Then $x_1 = \phi_n(\xi;\xi)$ differs from x_0 by no more than $M_n\eta^{n+2}/(n+2)!$, where $M_n = \max|\phi_n^{(n+2)}(x;\xi)|$ and $\eta = \max|x-\xi|$, and both maxima are for $a \leqq x \leqq b$. In terms of nth order determinants, it is shown how to pick $H_n(x;\xi)$ as a polynomial in x with coefficients which are rational functions of $f(\xi), f'(\xi), \cdots, f^{(n+1)}(\xi)$. A numerical example is given. It is not evident that the formulas have any advantage over those of Schröder [Math. Ann. 2, 317–363 (1870), p. 327], which are simpler to compute.

G. E. Forsythe (Los Angeles, Calif.).

Goldstine, Herman H., and von Neumann, John. Numerical inverting of matrices of high order. II. Proc. Amer. Math. Soc. 2, 188–202 (1951).

In a previous paper [Bull. Amer. Math. Soc. 53, 1021–1099 (1947); these Rev. 9, 471] the authors have considered a procedure for obtaining a matrix A^- intended to approximate the inverse of a matrix A of order n such that either A is practically singular or a bound C can be given for the matrix $AA^- - I$. This procedure is intended for digital computation on an automatic computer and consequently the discrepancy matrix $AA^- - I$ is dependent on the round off error e made in each step of the process. For quantities scaled to have their maximum less than one, e is less than $.5\beta^{-s}$ where β is the radix and s the number of places used. In the previous paper, a maximum for the bound of the matrix $AA^- - I$ was obtained by maximizing the effect of the round off errors. In the present paper these errors are treated as chance variables and a certain possibility which has a probability of less than .001 is ignored. Consequently, C is greatly improved. Thus if A is positive definite and λ and μ are its greatest and least characteristic roots, respectively, then the previous estimate of C, $14.24(\lambda/\mu)n^2\beta^{-s}$, becomes $114(\lambda/\mu)n\beta^{-s}$ which is clearly better for large n. The condition that A be not practically singular, which was $\mu > 10n^2\beta^{-s}$ before, is now lightened to $\mu > 80n\beta^{-s}$. If A is arbitrary, C is changed from $36.58(\lambda_I/\mu_I)^2n^2\beta^{-s}$ to $293(\lambda_I/\mu_I)^2n\beta^{-s}$ where λ_I and μ_I are the greatest and least characteristic roots of $(A^*A)^{\frac{1}{2}}$, respectively. There is an analogous change in the "nonsingularity" condition also. Logically, the present paper can be considered in three parts. In the first part a matrix T is considered with elements a_{ij} which are independent chance variables, normally distributed and with the same dispersion σ. For these, the distribution of the characteristic roots of $B = T^*T$ have been worked out independently by R. A. Fisher and P. L. Hsu [cf. Wilks, Mathematical Statistics, Princeton University Press, 1943, pp. 260–267; these Rev. 5, 41] and these results are used by the authors to obtain overestimates for the probability distribution function for the maximum of these roots which is also the bound of B. The second and most intricate part of the paper consists in the derivation of a relationship between the above case in which T has normally distributed elements and the corresponding results for a matrix T' having elements with the probability distribution of the actual discrepancy matrix $AA^- - I$, whose elements are formed by a process involving a variable number of additions of evenly distributed chance variables. The third part is concerned with the relatively minor variations of their previous discussion which are necessary in the present case. Utilizing various probability assumptions, the authors conclude that a machine having 27, 33, and 40 binary places would be adequate for systems having $n = 19$, 86, and 400 unknowns respectively.

F. J. Murray (New York, N. Y.).

Baetslé, P. L. Systématisation des calculs numériques de matrices. Bull. Géodésique 1951, 22–41 (1951).

Arrangements, checks, and clear, detailed instructions for a computer are given for the following familiar matrix operations: the resolution of any square matrix a into the product $h \cdot g$ of two triangular matrices; getting the determinant of a; getting a^{-1} (as $g^{-1} \cdot h^{-1}$); solving a system of linear equations $a \cdot x = l$ (using the triangular resolution); and the corresponding operations when a is symmetric, including the Doolittle and Cholesky methods for getting x. The matrix product $h \cdot g$ is performed essentially by superimposing h' and g and multiplying column by column. The method for getting a^{-1} and x for unsymmetric a is a "below-the-line" arrangement of the "unsymmetrical Cholesky" method of A. M. Turing [Quart. J. Mech. Appl. Math. 1, 287–308 (1948); these Rev. 10, 405]. *G. E. Forsythe.*

Banachiewicz, T. Résolution d'un système d'équations linéaires algébriques par division. Enseignement Math. 39 (1942–1950), 34–45 (1951).

Restated with matrices, this is an exposition, dated 1941, of the "unsymmetrical Cholesky" method of solving a system $Ax = b$ (A nonsingular) by means of the triangular resolution $A = LU$. Definitions: Row r of matrix B "extinguishes" column s if $b_{rs} \neq 0$ but $i > r$ implies $b_{is} = 0$; B is "elementary" if every row extinguishes at least one column. In the addenda, dated 1948, is a theorem: Every nonzero m-by-n matrix $A = BC$, where C and the transpose of B are elementary. Systems with singular A can be treated by the theorem. The author uses his own cracovians (tableaux with column-by-column multiplication) instead of matrices, and gives references to their use. *G. E. Forsythe.*

Abramov, A. A. On a method of acceleration of iterative processes. Doklady Akad. Nauk SSSR (N.S.) 74, 1051–1052 (1950). (Russian)

For the system $x_i = \sum_{j=1}^n a_{ij}x_j + b_i$, $i = 1, 2, \cdots, n$, the iterative process $x^{(2k+1)} = Ax^{(2k)} + b$, $x^{(2k+2)} = 2(Ax^{(2k+1)} + b) - x^{(2k)}$ is considered. It leads to a scheme in which ordinary iteration $x^{(k+1)} = Ax^{(k)} + b$ is applied k times with A, then k times with $A_1 = 2A^2 - 1$, then with $A_2 = 2A_1^2 - 1$, etc. A_1 has the same eigenvectors as A and its eigenvalues are given by ${}_{(1)}\lambda_i = 2\lambda_i^2 - 1$, etc. A limit to the error in the components of the eigenvector e_i is determined in terms of k if $|\lambda_i| < 1/\sqrt{3}$. Application to a system of 130 normal equations is briefly reported. *R. Church* (Annapolis, Md.).

Afriat, S. N. An iterative process for the numerical determination of characteristic values of certain matrices. Quart. J. Math., Oxford Ser. (2) 2, 121–122 (1951).

Let $A_{\nu\nu}$ be the principal square submatrix obtained by omitting the νth row and column from the Hermitian matrix a of finite order. Write $a(\lambda) = a - \lambda I$, $A_{\nu\nu}(\lambda) = A_{\nu\nu} - \lambda I$; let $a_{(\nu}, a_{\nu)}$ be respectively the νth row and column of a with their νth elements omitted. Under hypotheses which insure that $a_{\nu\nu}$ is sufficiently separated from the $a_{\mu\mu}$ ($\mu \neq \nu$), the author shows: (i) that λ_ν, a certain one of the characteristic values of a, satisfies the equation (*) $a_{\nu\nu} - \lambda = a_{(\nu}A_{\nu\nu}^{-1}(\lambda)a_{\nu)}$; and

(ii) that the following algorithm yields numbers $\lambda_r^{(n)}$ which converge to λ_r as $n\to\infty$: let $\lambda_r^{(1)}=a_{rr}$ and

$$a_{rr}-\lambda_r^{(n+1)}=a_{(r}A_{rr}^{-1}(\lambda_r^{(n)})a_{r)}.$$

Reviewer's note: the equation (*) is an elegant form of the "forward escalator" equation for the characteristic values of a [J. Morris, The Escalator Method . . . , Wiley, New York, 1947, p. 116, formula (17); these Rev. 9, 382].

G. E. Forsythe (Los Angeles, Calif.).

Morris, J. An application of the escalator process. Solution thereby of quasi-Hermitian frequency equations encountered in specific practical problems. Aircraft Engrg. 23, 136–137 (1951).

The author applies his "escalator process" for the solution of Lagrangian frequency equations to equations of Hermitian and quasi-Hermitian form. An orthogonal modal relationship for Hermitian equations is given. The author states that equations have been encountered in such problems as the whirling of contra-rotating propeller systems.

S. Levy (Washington, D. C.).

Hulthén, Lamek, and Laurikainen, K. V. Approximate eigensolutions of $(d^2\phi/dx^2)+[a+b(e^{-x}/x)]\phi=0$. Rev. Modern Physics 23, 1–9 (1951).

Der Grundgedanke des hier benützten Näherungsverfahrens besteht im folgenden: Um die äusserst mühsame Auflösung algebraischer Gleichungen höherer Ordnung, wie sie beim Ritz'schen Verfahren nötig ist, zu vermeiden, wird (ohne Beweis) folgendes Iterationsverfahren benützt: Man gewinnt zunächst aus einem n-gliedrigen Ansatz nach Ritz einen Näherungswert für den kleinsten Eigenwert und verwendet ihn, indem man die erste Gleichung für die Ermittlung der Koeffizienten weglässt, in einem $(n+1)$-gliedrigen Ansatz zur Bestimmung der Verhältniszahlen der Koeffizienten. Aus den ermittelten Werten bestimmt man den Quotienten der beiden in der Methode von Ritz vorkommenden quadratischen Integralausdrücken und benützt diesen Wert als verbesserte Annäherung für den Eigenwert. Dieser Grundgedanke lässt sich in einer modifizierten Form auch zur Berechnung von höheren Eigenwerten verwenden. Für die Ermittlung besonders grosser Eigenwerte werden ein zweites und andeutungsweise auch ein drittes Verfahren vorgeschlagen. Die vorliegende Differentialgleichung wird durch die Substitution $\Phi(x)=\exp[-(-a)^{\frac12}\cdot x]\omega(\xi)$, $\xi=1-e^{-x}$ auf die Form

$$\tau(1-\xi)\frac{d^2\omega}{d\xi^2}-(1-\tau)\frac{d\omega}{d\xi}-\frac{b\tau}{\ln(1-\xi)}\omega=0$$

gebracht, wobei $\tau=1/[2(-a)^{\frac12}+2]$ als kleiner Parameter behandelt wird.

P. Funk (Wien).

Corbett, James P. Electrical analogs of linear systems. Elec. Engrg. 68, 1075 (1949).

The above title is misleading and the paper is actually a digest of another paper entitled "Summary of transformations useful in constructing analogs of linear vibration problems" [Trans. Amer. Inst. Elec. Engrs. 68_1, 661–664 (1949)]. The latter title almost completely describes the paper; the author's concern is with the matrix transformations which lead to simplification in the construction and use of vibration analogs. *S. H. Caldwell.*

Bruce, V. G. A graphical method for solving vibration problems of a single degree of freedom. Bull. Seismol. Soc. America 41, 101–108 (1951).

A phase-plane diagram is used to solve first order equations which are piece-wise linear; and nonlinear equations are solved by approximation with piece-wise linear equations. *P. W. Ketchum* (Urbana, Ill.).

Collatz, L. Das Mehrstellenverfahren bei Plattenaufgaben. Z. Angew. Math. Mech. 30, 385–388 (1950).

Das sogenannte Mehrstellenverfahren ist eine Verschärfung des gewöhnlichen Differenzenverfahrens zur Gewinnung von Näherungslösungen bei Differentialgleichungen einfacher Bauart, das so eingerichtet ist, dass der Fehler mit einer höheren Potenz der Maschenweite gegen Null konvergiert als beim gewöhnlichen Differenzenverfahren, ohne dass die Rechenarbeit wesentlich erhöht wird [vgl. Collatz, Eigenwertaufgaben mit technischen Anwendungen, Akademische Verlagsgesellschaft, Leipzig, 1949; diese Rev. 11, 137]. Die vorliegende Arbeit enthält die Anwendung auf die Plattengleichung und zwar sowohl für den Fall der Belastung mit einer Gleichlast, als auch für die Gewinnung des Eigenwertes der Schwingenden Platte. Ausser quadratischen Netzen werden auch noch Dreiecksnetze betrachtet. Als Anwendungsbeispiel wird auch die Differentialgleichung für eine kreuzweise bewehrte Eisenbetonplatte herangezogen.

P. Funk (Wien).

Czetwertyński, E. Remarks on the adaptation of the equations of theoretical hydromechanics to practical computations. Arch. Méc. Appl., Gdańsk 2, 203–233 (1950). (Polish. Russian summary)

Wagner, Carl. On the solution of Fredholm integral equations of second kind by iteration. J. Math. Physics 30, 23–30 (1951).

L'auteur résoud l'équation $\varphi(x)=f(x)-L\int_a^b K(x,z)\varphi(z)dz$ par la formule d'itération $\psi^{(n+1)}(x)=\psi^{(n)}(x)+k^{(n+1)}(x)$, $k^{(n+1)}(x)=c(x)[\psi^{(n)}(x)+L\int_a^b K(x,z)\psi^{(n)}(z)dz-f(x)]$, où $c(x)$ est une fonction à choisir convenablement. L'auteur suggère $c(x)=-1/(1+L\int_a^b K(x,z)dz)$. Il donne un critère de convergence. En probique, on remplacera l'intégrale par une somme finie. On est ramené à la résolution par itération d'un système d'équations du premier degré. En s'inspirant des méthodes de von Mises et de Seidel diverses variantes sont proposées. Un exemple est traité numériquement.

J. Kuntzmann (Grenoble).

Malorov, F. V. The electrical representation of functions. Èlektričestvo 1950, no. 11, 51–57 (1950). (Russian)

The editor of Èlectričestvo points out that the work of the author in constructing universal function generating potentiometers, briefly abstracted in Izvestiya Akad. Nauk SSSR, Otd. Tehn. Nauk 1947, 1576–1578 (1947), anticipated the similar work of G. A. Korn [Rev. Sci. Instruments 21, 77–81 (1950); these Rev. 11, 401]. The potentiometers described in the present paper consist of about 8000 turns of very fine enamel covered constantan wire wound on an aluminum ring of 150 mm diameter. The brush is vernier controlled. When suitably chosen resistances are shunted across forty adjustable contacts which are provided, $f(x)$ is produced approximately. If these shunt resistances are themselves properly determined function generating potentiometers, all controlled from one y shaft, $f(x,y)$ is approximately represented. A circuit is shown and discussed for $f(x,y,z)$. It includes a linear potentiometer divided into n sections each shunted by a function generating potentiometer positioned by a servo. Each of the servos is controlled by a network such as already described for generating a function of two variables. As an example of accuracy attained, the author describes a nonadjustable potenti-

ometer for the sine function. It is 80 mm in diameter, non-uniformly wound, and is credited with less than .03% error due to use of 4 shunt resistances near 90° and the same number near 270°. *R. Church* (Annapolis, Md.).

Bell, J. Some aspects of electrical computing. I. Electronic Engrg. **23**, 213–216 (1951).

Allais, Julien. Un appareil utilisant la série Renard. C. R. Acad. Sci. Paris **232**, 1997–1999 (1951).

A convenient method of consulting large quantities of tabulated data is described. The data, which are arranged in cycloidal strips across a roll of photographic film, is read through a set of windows in a controllable motor-driven rotating disc. The rotation of the disc is geared to the translation of the film. *M. Goldberg.*

Babister, A. W., Marshall, W. S. D., Lilley, G. M., Sills, E. C., and Deards, S. R. The use of a potential flow tank for testing axi-symmetric contraction shapes suitable for wind tunnels. Coll. Aeronaut. Cranfield. Rep. no. **46**, 15 pp. (14 plates) (1951).

Meyer zur Capellen, Walther. Über die Koppelkurven des Zwillingkurbeltriebes. Z. Angew. Math. Physik **2**, 189–207 (1951).

The lobes of the lemniscoid curves traced by points on the connecting-rod of the symmetric anti-parallel four-bar linkage have been considered as wing profiles. The polar equations of these curves are derived using the node as origin. Radii of curvature and tangents at special points, the inscribed and circumscribed circles, the length of the chord and the thickness of the wing profile are obtained analytically and graphically. *M. Goldberg.*

Ruderman, Harry D. An extension of the nomogramic instrument. Proc. Amer. Math. Soc. **2**, 262–263 (1951).

A function f of n variables can be represented by a nomogram in Euclidean space of $n-1$ dimensions if it equals an $n\times n$ determinant whose ith row consists of elements V_{ij} involving the single variable s_i. The transversal is a hyperplane of $n-2$ dimensions. The present paper remarks that $V_{ij}=g_j(R_{i1},\cdots,R_{im})$, where R_{ik} is a function of the single variable s_i, for some integer m not exceeding n, since $m=n$, $g_j(u_1,\cdots,u_n)=u_j$, $R_{ik}=V_{ik}$ is one possibility. If $T_1,\cdots,T_m$ are m arbitrary functions of a single variable having inverses, the family $\sum_{j=1}^{n}A_jg_j[T_1(x_1),\cdots,T_m(x_m)]=0$ of hypersurfaces in $m-1$ dimensions is called by the author a nomogramic instrument which reduces to the usual transversal when $m=n$, $g_j=u_j$, $T_k=x_k$ and which can theoretically be used to construct the zeros of f. If $m<n$, the instrument has fewer than $n-2$ dimensions.

J. M. Thomas (Durham, N. C.).

Jecklin, H., und Zimmermann, H. Ergänzende Bemerkungen zur Reserveberechnung auf Basis hyperbolischer Interpolation (F-Methode). Mitt. Verein. Schweiz. Versich.-Math. **51**, 37–52 (1951).

Description of numerical arrangement and discussion of applicability of the method for approximate computation of grouped life insurance premium reserves treated earlier [same Mitt. **50**, 133–140 (1950); these Rev. **12**, 57].

H. M. Schaerf (St. Louis, Mo.).

ASTRONOMY

Fricke, A. Entfernungsmittelwerte bei der Ellipse. Z. Angew. Math. Mech. **31**, 181–185 (1951).

Mean values for the radius of an ellipse are derived with the center as origin and with one of the foci as origin for each of the three cases: (1) constant angular velocity, (2) constant areal velocity, (3) constant linear velocity in the orbit. *D. Brouwer* (New Haven, Conn.).

Sauvenier-Goffin, E. Note sur les pulsations non-radiales d'une sphère homogène compressible. Bull. Soc. Roy. Sci. Liège **20**, 20–38 (1951).

The author studies nonradial oscillations of a homogeneous but compressible gas sphere under the influence of its own gravity. In the first part of her paper, the differential equations of the problem are set up, and the author points out that the neglect of the disturbance of the gravitational potential invoked by the oscillations leaves the equations formally unchanged. The numerical results for the case of a spherical harmonic distortion of order $n=2$ having already been obtained by Pekeris [Astrophys. J. **88**, 189–199 (1938)], the author proceeds to extend Pekeris' work for three lowest overtones corresponding to the harmonics $n=2$, 3, and 4. For real values of the characteristic frequencies of oscillation, the number of zeros of the pressure variation is found to exceed by one that of the zeros of the corresponding density variation, while for imaginary frequencies of oscillation, these two numbers are equal. If the disturbance in gravity is ignored, the corresponding error in the frequency of oscillation becomes a maximum (about 15%) for the fundamental mode of $n=2$, and diminishes with increasing n as well as the increasing mode; while the corresponding error in the computed values of the pressure disturbances is shown to be almost negligible.

Z. Kopal (Manchester).

Frank-Kameneckiĭ, D. A. Oscillatory stability and the auto-oscillations of stars. Doklady Akad. Nauk SSSR (N.S.) **77**, 385–388 (1951). (Russian)

The author introduces effective temperature coefficients and assuming an increase in the relative amplitude of pulsation from the surface to the center of the star and the basic source of energy to be the hydrogen cycle finds a condition of oscillatory instability in the presence of low central temperatures and finite limit of amplitude of oscillation. *R. G. Langebartel* (Urbana, Ill.).

Ledoux, P. Contributions à l'étude de la structure interne des étoiles et de leur stabilité. Mém. Soc. Roy. Sci. Liège (4) **9**, 3–294 (1949).

This book, a thesis from Université de Liège, summarizes on a broad background the results of the author's investigations in his special field, the study of stellar structure. The first chapter contains a general survey of the three main branches of astrophysics, stellar atmospheres, interstellar matter and the interiors of stars. The next chapter deals with stellar models with varying chemical composition, following up the idea of Hoyle and Lyttleton that discontinuities in the mean molecular weight may arise in the outer layers of a star through aggregation of interstellar hydrogen. If the ratio μ_i/μ_e of the molecular weights μ_i in the interior and μ_e in the outer layers is below a critical value nothing will happen. But as soon as this critical value is reached,

an instability will arise. The author shows in contradiction to previous investigations how this instability finally leads to the formation of a transition zone in equilibrium. This adjustment takes place in a short time, and will have only a very slight influence on the luminosity and radius of the star. Since diffusion in a star is a very slow process a difference in chemical composition develops between the convective core and the rest of the star, caused by exhaustion of lighter elements due to the energy-production process. This case is more difficult than the previous one, and can not be handled analytically. It is shown how it is possible to find the variation of the physical parameters in the transition zone, which will appear in this case too, when the ratio exceeds a critical value much smaller here than for discontinuities in the outer layers. By using the boundary conditions at the discontinuity surface a unique connection with the convective core is obtained. The theory is applied to numerical examples. If, as in the case of the sun, luminosity, radius, and mass are given quantities, a relation will exist between the central temperature and the molecular weights in the two parts of the star. It seems to be impossible to get agreement with observations in the case of the sun without postulating a considerable admixture of helium. With hydrogen and helium contents respectively 0.56 and 0.40 in the radiative part, the author finds the corresponding values for the convective core of the sun to be 0.26 and 0.70. Assuming the initial composition to be uniform it is possible in this way to give an estimation of the age of the sun. The resulting age of $5\cdot 10^9$ years is in good accord with values found by other methods. This theory also makes it possible to find the path taken by the sun in the Hertzsprung-Russell diagram during its evolution.

The third chapter contains a discussion of the dynamical stability of nonrotating stars. First the equation for a general adiabatic perturbation is derived. In the question of the dynamical stability of stars the ratio of specific heats for a mixture of matter and radiation Γ_1, is the quantity of main importance. Fowler and Guggenheim's expression for Γ_1 when ionization is at work is developed. In a following section the author considers the relation between the stability with regard to nonradial perturbations and the stability against convection currents (Schwarzschild's criterion). Neglecting the perturbations in pressure and gravity potential, the stability condition found is identical with that of Schwarzschild. If the conditions mentioned are not satisfied, it is difficult to arrive at a general conclusion regarding the stability.

In the following chapter radial pulsations of a star are discussed. The author shows how to obtain an approximate expression for the frequency of the fundamental oscillation using the theorem of the virial. From this expression it is possible to get an idea of the physical conditions influencing stability. As in the case of the homogeneous model the stability depends on the value of the ratio of specific heats. A certain mean of this quantity over the star must be greater than 4/3. To get an idea of the effects of regions with Γ_1-values smaller than 4/3 several special models are treated numerically. The pulsation equation is solved at the limit of instability (frequency equals zero), first in the case of two constant values for Γ_1 in the star. The investigation gives the position of the limit between the two domains when the star is on the verge of instability. These calculations are carried out for the homogeneous model (mass concentration $=1$) and for the standard model (mass concentration $=54$). The effect of zones where Γ_1 drops below 4/3 is also considered. Applications to real stars show them to be well within the limit of stability, with the exception of stars with very large masses. In this case special assumptions regarding the chemical composition must be made to explain their stability. The fifth and last chapter deals with the dynamical stability and pulsations of rotating stars. An expression for the pulsation frequency of a homogeneous star is first found for pulsations conserving the axial symmetry, showing the stabilizing effect of rotation. The frequency is also found for a homogeneous star maintaining nonradial pulsations when the perturbations are represented by spherical harmonics. When extending the investigations to stars with an arbitrary density distribution serious difficulties have to be overcome. But by using the theory of the virial the author obtains a generalized expression for the frequency, showing the rotation to increase the stability in this case too. At the end of each chapter there is an extensive list of references.

E. Jensen (Oslo).

Bagge, Erich. Eine Deutung der Expansion des Kosmos. Z. Physik **128**, 239–254 (1950).

The author develops the novel suggestion that Hubble's law, $v=\alpha R$, correlating the apparent velocities of recession v of the nebulae with their respective distances R, can be derived, with a good estimate for α, from a simple extension of Fermi's theory of the origin of cosmic ray energies by interstellar magnetic fields [Physical Rev. (2) **75**, 1169–1174 (1949)]. The derivation falls into two parts. First, given the phenomenon of expansion, the author develops the idea (which, although he does not say so, is not original) that a law of Hubble's type, with a good estimate for α, can be derived for a simple Newtonian homogeneous world-model in which every particle (nebula) has the appropriate velocity of gravitational escape. The author refines the calculation by appealing to restricted relativity and deduces a corrected law of the form $v/c=\alpha R/c-(1/20)(\alpha R/c)^3+\cdots$. In the second part, the author endeavours to account for the expansion by the novel suggestion that, in some cases of collision between two protons in interstellar space, it can happen that both acquire energies above 10^8 eV, the "injection-energy" required for Fermi's process to operate, even when only one had such an energy originally. He develops a "diffusion" equation for this process and shows that when the world-radius is less than a certain critical value the Fermi mechanism will lead to a cosmic ray intensity increasing exponentially in time. It is suggested that when the nebulae were originally closely packed together an explosive process of this type gave rise to world-expansion.

G. J. Whitrow (London).

RELATIVITY

Udeschini, Paolo. Le equazioni di prima approssimazione nella nuova teoria relativistica unitaria di Einstein. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 9, 256–261 (1950).

The author considers to a first approximation the field-equations, in empty space, of Einstein's new unified theory [The Meaning of Relativity, rev. ed., Princeton University Press, 1950; for a review of the 1945 ed. see these Rev. 7, 87]. He takes the fundamental tensor to be $g_{ik}=a_{ik}+b_{ik}$, where the a_{ik} have Galilean values and the b_{ik} are small and in general asymmetric. He finds that, to the first order, the gravitational and electromagnetic fields separate out, the

field-equations splitting up into two sets. One of these sets is the same as that obtained to a first approximation from the ordinary relativity theory of gravitation, and the other is the set of Maxwell's equations for regions free from charge. These results hold to a first approximation only, for even in the second approximation the gravitational and electromagnetic fields prove to be indissolubly linked, as the author proposes to show in a later paper. *H. S. Ruse.*

de Castro Brzezicki, A. The relativistic theories of Einstein. Gaceta Mat. (1) 3, 18–27 (1951). (Spanish)
Expository paper.

Taub, A. H. Empty space-times admitting a three parameter group of motions. Ann. of Math. (2) 53, 472–490 (1951).

In this paper the author discusses the equation $R_{\mu\lambda}=0$ (Ricci tensor) for space-times admitting a 3-parameter group of motions with generators $\xi_a{}^h$, which are space-like. If the matrix $M=\|\xi_a{}^h\|$ is of rank 3, one may introduce a coordinate system for which $ds^2=(dx^0)^2-h_{ij}dx^idx^j$ $(i,j=1,2,3)$. For each of the nine types of transitive 3-parameter groups, the solution of the Killing equations is discussed. Then it is shown that the equation $R_{\mu\lambda}=0$ leads to a consistent system of ordinary first order differential equations for h_{ij}. For most of the types the $g_{\mu\lambda}$ have spatial singularities. An example is given where the $g_{\mu\lambda}$ are finite for all finite values of the "time", $R_{\mu\lambda}=0$, whereas $R_{\nu\mu\lambda}^{\cdots h}\neq 0$. Whether this example contradicts Mach's principle cannot be decided as long as a definition of an "essential" singularity is lacking in the theory. Finally the case that M is of rank 2 is completely discussed. *J. Haantjes* (Leiden).

Kranjc, Aldo. Proprietà gruppali del tensore energetico. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 483–486 (1950).

Kranjc, Aldo. Proprietà gruppali del tensore energetico. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 8, 578–582 (1950).

A discussion of the significance of the general-relativity field equations $G_{ik}-\frac{1}{2}g_{ik}(G-2\lambda)=-\beta T_{ik}$ leads the author, in the first of these papers, to lay down the "principle of isomorphism", namely that the group of physical operations performed on a given system is isomorphic with the group of transformations induced in the space-time S_4 determined by that system, This, he says, is really no more than a precise form of the principle of the geometrization of physics [McVittie, Cosmological Theory, Methuen, London, 1937, p. 39]. An application of this principle leads him in the second paper to consider groups of motions into themselves of space-times of the form $ds^2=V^2dt^2-\sum_{1,2,3}a_{rs}dx_rdx_s$ (V constant, a_{rs} functions of x_1, x_2, x_3), with physical interpretations. *H. S. Ruse* (Leeds).

Pratelli, Aldo M. Sul campo elettromagnetico "ortogonale" nello spazio-tempo. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 9, 251–256 (1950).

Pratelli, Aldo M. Sul campo elettromagnetico "ortogonale" nello spazio-tempo. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 9, 331–336 (1950).

The author considers the electromagnetic 6-vector $F_{\alpha\beta}$ and the potential 4-vector ϕ_α in the space-time S_4 of special relativity, in the case when the skew tensor $F_{\alpha\beta}$ is simple, that is, when its fundamental invariant is zero. The electric and magnetic 3-vectors **E**, **H** are then orthogonal (conventionally so if either is zero). In this case $F_{\alpha\beta}$ is expressible in terms of two potential functions f, g by the formula

$$F_{\alpha\beta}=\frac{\partial f}{\partial x^\alpha}\frac{\partial g}{\partial x^\beta}-\frac{\partial f}{\partial x^\beta}\frac{\partial g}{\partial x^\alpha}.$$

The main result of the first part of the paper is contained in the theorem: The characteristic condition that the differential form $\phi_\alpha dx^\alpha$ should admit an integrating factor is that $F_{\alpha\beta}$ should be simple. The result is discussed in terms of the geometry of S_4.

The second part of the paper is mainly concerned with the "work" $L\equiv\int_\Gamma F_{\alpha\beta}[dx^\alpha, dx^\beta]$, [ ] denoting the exterior product, of the tensor $F_{\alpha\beta}$ over a given nonclosed 2-surface Γ in S_4. If f, g are interpreted as rectangular Cartesian coordinates in a plane, then L is equal to a certain area in that plane. Moreover, L is zero on every open 2-surface wholly contained in a "potential hypersurface" $\Phi(f,g)=0$. The intersection Ω of two such hypersurfaces is an equipotential surface as defined by Whittaker. Lastly, when the charge-and-current 4-vector J^α is zero, the tensor $F_{\alpha\beta}$ is solenoidal as well as irrotational, and Whittaker's magnetopotential surfaces appear in a manner similar to that indicated above. *H. S. Ruse* (Leeds).

Galli, Mario. Considerazioni sul II postulato della relatività. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 9, 262–267 (1950).

The author begins with the remark that special relativity is based upon two postulates, of which the first asserts that the laws of nature are independent of the particular inertial frame used, and the second that the velocity of light is constant. For various reasons the second postulate is less well established than the first, and in any case possesses the demerit of being based upon a very particular physical phenomenon. Now the Lorentz transformation $x=(x'+vt')/(1-\alpha v^2)^{\frac{1}{2}}$, etc., is in accord with the first postulate, the second being needed in order to make the identification $\alpha=1/c^2$. The author discusses various ways, based on different experimental situations, whereby this identification could be made without invoking the second postulate. His final remark is that the Lorentz transformation can be justified without an anticipatory guarantee of the constancy of the velocity of light, and that it must be accepted even if experiments designed for the purpose are regarded as uncertain. *H. S. Ruse* (Leeds).

Galli, Mario. L'induzione elettromagnetica e il principio di relatività. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 9, 343–348 (1950).

Cullwick [Nature 161, 969–970 (1948)] has observed that, if an electric charge in uniform motion passes near a stationary closed conducting loop, an E.M.F. and current should be induced. If, however, the loop moves and the charge is stationary, the relative motion being the same as before, the Maxwell-Lorentz theory gives zero induced E.M.F. In the paper under review the author remarks that this may give rise to doubts whether the ordinary laws of electromagnetism are in complete harmony with the principle of relativity, and he proceeds to discuss the matter in some detail. In the former case (fixed circuit, assumed plane, charge moving with velocity v in the same plane), he finds a formula for the induced E.M.F., which proves to be of the second order in v/c, and remarks that, as the Maxwell-Lorentz electrodynamics is relativistic only if second-order

terms are neglected, it is not surprising if this dynamics gives zero E.M.F. in the inverse case. A closer investigation from the point of view of relativity leads him to the conclusion that there is in fact no paradox, and that any doubt about the inner coherence of relativistic electrodynamics is without foundation. *H. S. Ruse* (Leeds).

McCrea, W. H. Relativity theory and the creation of matter. Proc. Roy. Soc. London. Ser. A. **206**, 562–575 (1951).

It is shown that the continuous creation of matter can be treated by the standard methods of general relativity provided that the possibility of a "zero-point" stress is allowed. The matter "created" is the mass-equivalent of the work done by this stress in the expansion of the universe. According to relativity theory this stress also makes a contribution to the density of gravitational mass. When these two effects are taken into account it is shown that the Newtonian analogues of relativistic models of the universe discovered by Milne and the writer can be extended to cover all relativistic models. This result makes it possible to obtain a unified physical interpretation of Hoyle's treatment of the cosmological problem. It is suggested that the interpretation in terms of a zero-point stress may lead to a connexion with the quantum theory of fields. *Author's summary.*

Milne, E. A. Gravitation and magnetism. Monthly Not. Roy. Astr. Soc. **110**, 266–274 (1950).

It is shown by the methods of kinematic relativity that there should be a connection between gravitation and magnetism of the type suggested by the empirical formulae of Blackett and Wilson, multiplied however by certain dimensionless ratios. The field of a rotating system cannot be represented by a dipole. The derivation only applies rigorously to a rotating system like a galaxy, with its centre at a fundamental particle. The results obtained suggest that for a given mean density and given angular momentum, a highly flattened system should have an effective magnetic moment greater than a less flattened system. *Author's summary.*

Graef Fernández, Carlos. Principles of conservation in the theory of gravitation of Birkhoff. Bol. Soc. Mat. Mexicana **5** (1948), 7–14 (1950). (Spanish)

L'auteur étudie, en théorie de la gravitation de Birkhoff, les intégrales premières du mouvement d'un masse d'épreuve (planète) dans le champ d'une masse fixe (soleil); M_0 désignant la masse du soleil, M_p la masse au repos de la planète, r le rayon-vecteur, (x, y) les coordonnées cartésiennes de la planète et v sa vitesse, il obtient les intégrales premières

$$-M_0M_p/r+\tfrac{1}{2}M_p\log(1/(1-v^2))=E,$$

$(M_p/(1-v^2))(x\dot{y}-y\dot{x})=H$, qui généralisent immédiatement les intégrales premières classiques. *A. Lichnerowicz* (Paris).

García, Godofredo. The foundations and the construction of a new theory of general relativity. The concept of time. The new complete law of universal gravitation. The differential equations of motion of the new dynamics. Actas Acad. Ci. Lima **14**, 3–41 (1951). (Spanish)

Un nouvel exposé de la théorie relativiste de la gravitation de Birkhoff en s'efforçant de recourir à des processus constructifs et avec quelques légères modifications. *A. Lichnerowicz* (Paris).

Coxeter, H. S. M., and Whitrow, G. J. World-structure and non-Euclidean honeycombs. Proc. Roy. Soc. London. Ser. A. **201**, 417–437 (1950).

This paper is mainly concerned with the problem of finding all the uniform honeycombs in hyperbolic 3-space. There are many more than the four given by Schlegel in 1883; one of the new honeycombs is related to the set of integral solutions of the Diophantine equation $t^2-x^2-y^2-z^2=1$. An interesting by-product is a simple set of generators for the group of all integral Lorentz transformations [cf. A. Schild, Canadian J. Math. 1, 29–47 (1949); these Rev. 10, 579]. The relativity problem which gave rise to the questions discussed here is that of finding a world model of discrete particles (nebulae) satisfying a suitably modified form of the Cosmological Principle. The conclusions are that all the uniform honeycombs are too coarse to be of direct cosmological significance; the observed nebulae are too close together. This means that, if hyperbolic space is in fact the appropriate space for a world model, then some irregularity in the distribution of nebulae is geometrically inevitable. *A. G. Walker* (Sheffield).

MECHANICS

✶Hrones, John A., and Nelson, George L. Analysis of the Four-Bar Linkage. Its Application to the Synthesis of Mechanisms. The Technology Press of The Massachusetts Institute of Technology; John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1951. xx+730 pp. $15.00.

This monumental work is an 11×17 in. atlas of motions of selected points of the connecting-rod planes of all the (146) crank-and-rocker linkages whose link lengths are multiples nc of one-half the crank length $2c$, with $n\leqq 8$ for the connecting-rod and the rocker. The motions are represented by means of dashed trajectories; the initial points of the dashes correspond with the drive-crank positions 5° apart. The accuracy claimed is 0.01 in. for trajectory traces, and 5 per cent in dash spacing. The apparatus used to trace the trajectories had a drive-crank length of 5 in., the tolerances being 0.005 in. The atlas is meant to be scanned for the approximate shape desired.

The selected points form (in the plane of the connecting rod A) a rectangular lattice of spacing c, bounded by the four lines parallel and perpendicular to A at distances c from the ends of A. If the length of A is nc, there will be five parallel rows of $n+5$ points, and a separate sheet is given to each row. Frequently some of the points on the rocker side are omitted. This inflexible selection is claimed to lead to an exhaustive survey of the complete performance and the characteristic motions of the subject linkages. With this, this reviewer cannot agree. If the connecting-rod gets large (say four times the crank length) the shapes may change so fast in the neighborhood of the rocker that a jump of c from one tracing point to another may skip the figure-eight curves (cf. pp. 556, 561, 696–699). On the crank side, the shapes may change so slowly that waste is involved in showing so many

curves. Significant changes may occur outside the rectangle shown (page 560 omits some triangle-shaped curves). In fact, at the risk of appearing captious and ungrateful, this reviewer expresses the regret that such monumental labor has not been rounded off by indicating the boundaries between regions whose points describe trajectories of different shapes. However, even in the present form the atlas will be found enormously useful in many design problems.

A. W. Wundheiler (Chicago, Ill.).

✶**Dimentberg, F. M. Opredelenie položenii prostranstvennyh mehanizmov. Primenenie metoda "vintov" k issledovaniyu peremeščenii prostranstvennyh mehanizmov. [The Determination of the Positions of Spatial Mechanisms. Application of the Method of "Screws" to the Investigation of the Displacements of Spatial Mechanisms].** Izdat. Akad. Nauk SSSR, Moscow, 1950. 142 pp.

This book is an expansion of a previous article [Trudy Sem. Teorii Mašin i Mehanizmov 5, no. 17, 5–39 (1948); these Rev. 12, 549]. The algebra of sliding vectors and finite rigid displacements based on Clifford's numbers, was sketched in the review cited. Here it occupies the first 40 pages. The following 60 pages are devoted to four-bar linkages with all the possible types of pairs at its joints. Forty pages are given to the equations of configuration and their solution, twenty to metric conditions for the existence of passive constraints reducing cylindrical pairs to purely rotary ones. A complete set of these conditions is given for four-bar linkages with cylindrical pairs only. Twenty-eight pages deal with five-bar linkages containing three rotary and two cylindrical pairs, or four rotary and one spherical pair (in several sequences). Equations for the existence of passive constraints are stated but not explicitly solved: Their complexity seems prohibitive. Throughout the book the mathematical labor, due to the sheer number of terms to consider, is huge in these problems (formulas spill over the page frames). This is probably a congenital feature of the problems in spatial linkages. *A. W. Wundheiler.*

van der Woude, W. The twofold signification of the point of Ball and the points of Burmester and some other points in the motion of the plane rigid system. Nederl. Akad. Wetensch. Proc. Ser. A. **54** = Indagationes Math. **13**, 117–122 (1951).

The points in a moving plane which, momentarily, trace straight lines lie on a circle called the inflection circle. At one of these points, called the Ball point, the path has contact of the third order with its tangent. The Ball point is also on the focal curve which is the locus of the points in the moving plane whose paths have contact of at least the third order with the circle of curvature. At four of the points on this curve, called Burmester points, the contacts are of the fourth order. It is shown analytically that the inflection circle touches its envelope in the moving plane in the Ball point and that it touches its envelope in the fixed plane in a point having properties similar to the Ball point. Hitherto, the latter point has gone unnoticed. (Shall we call it the van der Woude point?) Also, it is shown that the focal curve touches its envelope in the moving plane in the four Burmester points while it touches its envelope in the fixed plane in four newly-discovered corresponding points. This is an extension of a paper by Bottema [same Proc. **52**, 643–651 = Indagationes Math. **11**, 205–213 (1949); these Rev. **11**, 217].

M. Goldberg (Washington, D. C.).

Tišin, M. M. The design of cam mechanisms with a rocking crank follower. Akad. Nauk SSSR. Trudy Sem. Teorii Mašin i Mehanizmov **2**, 105–110 (1947). (Russian)

Let A be the fixed point of a cam, C that of a straight-line follower, K the point of tangency, $\phi = \angle ACK$, $\psi = \angle CKA$. The follower will rock only if $d\psi/d\phi < \frac{1}{2}$. Then CK and the curvature at K are determined in terms of ψ, $d\psi/d\phi$, and $d^2\psi/d\phi^2$. The condition of convexity is stated in the same terms, and applied to the limiting case of a sliding follower.

A. W. Wundheiler (Chicago, Ill.).

Ilić, Branislav. A graphical construction of the normal acceleration from the rotational velocity. Srpska Akad. Nauka. Zbornik Radova, Knj. **5**. Mašinski Institut, Knj. **2**, 115–126 (1950). (Serbo-Croatian)

A graphical construction of $a = v^2r^{-1}$ when v and r are given. *M. Golomb* (Lafayette, Ind.).

Ilić, Branislav. The construction of relative velocities in the plane motion of mechanisms. Srpska Akad. Nauka. Zbornik Radova, Knj. **5**, Mašinski Institut, Knj. **2**, 85–95 (1950). (Serbo-Croatian)

A graphical construction of the velocities of the points of a rigid rod when the velocities of the endpoints are given.

M. Golomb (Lafayette, Ind.).

Artobolevskiĭ, I. I. On some forms of the equations of motion of machinery aggregates. Doklady Akad. Nauk SSSR (N.S.) **77**, 977–979 (1951). (Russian)

Marsicano, Fénix R. Relation between the parameters of Lagrange and the constraints in a solid which rolls without slipping on a plane. Ciencia y Técnica **116**, 150–154 (1951). (Spanish. French summary)

The author discusses the usual treatment, involving the use of Lagrangean indeterminate multipliers, of the motion of a rigid body which rolls without slipping on a fixed plane. He derives an interpretation of the multipliers in terms of components of the force of constraint which the plane exerts upon the body. *L. A. MacColl* (New York, N. Y.).

Colombo, G. Sulle configurazioni di equilibrio di un velo flessibile ed inestendibile, sviluppabile. Rend. Sem. Mat. Univ. Padova **20**, 153–166 (1951).

The author takes up the researches of E. Laura [Atti Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. **99**, 339–356 (1940)] on Beltrami's theory of inextensible membranes. While Laura, using orthogonal coordinates, reduced the problem for a developable surface to a set of integro-differential equations, the author by using the device introduced recently by C. Tolotti [Giorn. Mat. Battaglini (4) **3**(79), 1–48 (1950); these Rev. **12**, 141] of choosing an oblique coordinate system particularly suitable to developable surfaces is able to reduce the problem to a set of ordinary differential equations of first to third order together with six algebraic equations. He concludes that the solution must contain 10 arbitrary constants. For the case of a quadrilateral with two opposite sides assigned to given positions in space, the author finds 12 conditions to be satisfied and hence concludes a solution cannot exist. For the case of a triangle with one side and the opposite vertex fixed, he concludes similarly that it is plausible or at least not impossible that a solution exists. *C. Truesdell.*

Hagiwara, Takahiro. On the periods of the vibration that is produced by superposing several simple harmonic vibrations of different period and amplitude. Bull. Earthquake Res. Inst. Tokyo **23**, 1–22 (1945). (Japanese. English summary)

"The theoretical investigation was done for the periods that appear when two or three simple harmonic motions are superposed. The calculations were made with a graphical method, and the frequencies of occurrence of the periods were found thereby." *Author's summary.*

Sezawa, Katsutada, and Kanai, Kiyoshi. On the initial movement of a seismograph subjected to an arbitrary earthquake motion, solved with operational calculus. I. Bull. Earthquake Res. Inst. Tokyo **19**, 162–176 (1941). (English. Japanese summary)

The authors take as starting point the approximate equation of small forced motion of a damped oscillator $\ddot{x}+2k\dot{x}+n^2x=-\ddot{\xi}$. The oscillator is taken to represent a seismograph pendulum and $\ddot{\xi}$ the acceleration of the ground on which the seismograph rests. To this system they apply methods of the operational calculus based on Mellin's inversion theorem using the Bromwich-Wagner integral in the complex plane to determine the first response of the pendulum to certain arbitrary forcing motions of the ground. They find that in all cases of instantaneous movement the first motion of the pendulum is opposite to that of the ground and that it consists of two parts, as in elementary solutions, the one corresponding to forced oscillation and the other to free oscillation. Particular cases considered are (1) ground at rest, (2) ground subjected to an isolated square wave, (3) ground suddenly displaced and then relaxed exponentially, (4) ground going from rest into uniform sinusoidal motion assuming various relative periods, (5) ground going from rest into damped harmonic motion. For the mathematical developments the original paper must be consulted. *J. B. Macelwane* (St. Louis, Mo.).

Sezawa, Katsutada, and Kanai, Kiyoshi. On the initial movement of a seismograph subjected to an arbitrary earthquake motion, solved with operational calculus. II. Bull. Earthquake Res. Inst. Tokyo **19**, 443–457 (2 plates) (1941). (English. Japanese summary)

In this second paper the authors discuss additional operational solutions for the response of a seismograph pendulum to prescribed motions of the ground. The cases considered are: (1) ground displacement starting from rest, increasing at a constant rate for a given interval of time, then leveling off at a constant value; (2) ground displacement starting from rest, increasing at a parabolic rate to a constant value; (3) alternating square wave; (4) alternating symmetrical saw tooth wave; (5) steadily increasing positive displacements followed by instantaneous return to zero; (6) instantaneous positive displacements followed by return at a constant rate. Model experiments to test the calculations were performed and fair agreement shown. The original paper must be consulted for the mathematical details.

J. B. Macelwane (St. Louis, Mo.).

Davies, E. T. J., and Mauranen, V. An application of Cornu's spiral to the mathematical theory of the motion of an unrotated rocket. Math. Gaz. **35**, 12–18 (1951).

The angular deviations of a fin-stabilized unrotated rocket from its mean trajectory due to a cross wind, initial cross spin, and malalinement of axial thrust can be estimated approximately in terms of three functions $E(v_0, v_1)$, $G(v_0, v_1)$, and $F(v_0, v_1)$ respectively, when certain assumptions such as constant acceleration are made. [See, for example, the reviewer, Philos. Trans. Roy. Soc. London. Ser. A. **241**, 457–585 (1949), §6.4; these Rev. **10**, 749.] Thus, for example,

$$E(v_0, v_1)=\int_{v_0}^{v_1} \sin \tfrac{1}{2}\pi(u^2-v_0^2)du \\ +\{1-\cos \tfrac{1}{2}\pi(v_1^2-v_0^2)\}/(\pi v_1).$$

Here v_0 and v_1 are dimensionless parameters proportional to the velocity at launch and burnt respectively. The purpose of the paper is to provide quick approximate methods of evaluating these functions. To do this Cornu's spiral $x=\int_0^v \cos \frac{1}{2}\pi u^2du$, $y=\int_0^v \sin \frac{1}{2}\pi u^2du$ is drawn and the curve calibrated with values of the parameter v. It is found that the three functions can be simply expressed in terms of certain easily measured lengths. Thus $E(v_0, v_1)$ is approximately $P_0N_0-P_1C$, where C is the point $(\frac{1}{2}, \frac{1}{2})$, P_i is the point on the curve with parameter v_i $(i=0, 1)$ and N_0 is the foot of the perpendicular from C to the normal at P_0. The error involved in the approximations should not exceed 2% in most practical applications. *R. A. Rankin.*

Hydrodynamics, Aerodynamics, Acoustics

Synge, J. L. On permanent vector-lines in N dimensions. Proc. Amer. Math. Soc. **2**, 370–372 (1951).

The author extends to a Euclidean space of N dimensions the following theorem of Zorawski [Anz. Akad. Wiss. Krakau **1900**, 335–342]: In order for the trajectories of the vector field $c_i(x, t)$ to be material lines with respect to a given velocity field $v_i(x, t)$ it is necessary and sufficient that $c_j(\partial c_i/\partial t+c_{i,k}v^k)+c_iv_{j,k}c^k$ be symmetric. [Reviewer's note. The theorem holds in an affine space, as may be seen by taking the proof for Euclidean 3-space given by Prim and the reviewer [same Proc. **1**, 32–34 (1950); these Rev. **11**, 696] and replacing each statement about the vanishing of a cross-product $a\times b$ by an obvious analogue about the symmetry of the dyadic product a_ib_j.] *C. Truesdell.*

Truesdell, C. Verallgemeinerung und Vereinheitlichung der Wirbelsätze ebener und rotationssymmetrischer Flüssigkeitsbewegungen. Z. Angew. Math. Mech. **31**, 65–71 (1951). (German. English, French, and Russian summaries)

Consider a continuous flow of fluid whose vortex lines are stationary and normal to stream surfaces $x_w=\text{constant}$. For the line element ds write $ds^2=h^2dx_w^2+\cdots$, and let v_0 be a function which is constant for a given particle so that $Dv_0/Dt=0$. Define vectors $\mathbf{v}_e=\mathbf{v}/v_0$, $\mathbf{w}_e=\text{curl } \mathbf{v}_e$, where $\mathbf{v}$ is the fluid velocity. Take scalar functions A and ν which satisfy the equations

$$\text{grad}\left(\frac{A}{v_0}\right)\times\frac{\partial \mathbf{v}_e}{\partial t}=\frac{\mathbf{w}_e}{v_0}\frac{\partial A}{\partial t},\quad \frac{1}{v_0}\frac{\partial \nu}{\partial t}+\text{div}\,(\nu\mathbf{v}_e)=0.$$

Then if it is possible to choose v_0 and A to satisfy the additional equation

$$\mathbf{w}_e\cdot\text{curl}\left\{A\left(\frac{1}{v_0}\frac{\partial \mathbf{v}_e}{\partial t}+\mathbf{w}_e\times\mathbf{v}_e\right)\right\}=0,$$

the function $Aw_e/(h\nu)$ is constant for a fluid particle. If we take $A=1$, $v_0=1$, $\nu=\rho$, this purely kinematical theorem includes and generalises known vortex theorems for plane

($h=1$) and axisymmetric flow (h=distance from the axis). It also leads to a new theorem in gas dynamics. For a gas in steady motion, with $\mathbf{w}\cdot\mathbf{v}=0$, in which the pressure p, the density ρ and the specific entropy η are connected by a relation of the form $\rho=H(\eta)/\Pi'(p)$, the author proves that $w_r d \log \Pi(p)/dp$ is constant along a streamline.

L. M. Milne-Thomson (Greenwich).

Rapoport, I. M. On the plane jet flow of an ideal fluid about a solid body. Ukrain. Mat. Žurnal 2, 107–117 (1950). (Russian)

The author considers a class of symmetric 2-dimensional free streamline flows which can be used to approximate other flows of this sort. The case of an obstruction in the form of an arc with end points as separation points and also the case of a closed curve with unknown separation points are considered. Let the z-plane (the physical plane) be mapped conformally onto the ζ-plane with a cut along the positive real axis so that the ends of the obstructing arc map onto the points $1\pm 0i$. Then $d\zeta/dz=Cv$, where $v=v_x-iv_y$ (the complex velocity) and C is a real constant. The author considers the flows obtainable from

$$v(\zeta)=\frac{v_\infty\zeta^{\frac{1}{2}}}{(\zeta-1)^{\frac{1}{2}}+i}\exp i[\zeta^{\frac{1}{2}}P(\zeta)-(\zeta-1)^{\frac{1}{2}}Q(\zeta)],$$

where $P(\zeta)$ and $Q(\zeta)$ are polynomials with real coefficients and related by $Q(\zeta)-(1-\zeta)^{\frac{1}{2}}P(\zeta)=O(\zeta^{-1})$ as $\zeta\to\infty$. With such a choice of v it is possible to satisfy the boundary conditions of the hydrodynamic problem, although the problem of the closed curve is somewhat more elaborate than the arc with end points. Approximations are carried out in the Z-plane, where $Z=Ke^{i\varphi}$, K the curvature and φ the angle between the tangent to the arc and the imaginary axis.

J. V. Wehausen (Providence, R. I.).

Peters, A. S. The effect of a floating mat on water waves. Comm. Pure Appl. Math. 3, 319–354 (1950).

L'auteur étudie le problème suivant: "La surface d'une masse d'eau de profondeur infinie est recouverte dans un demi-plan par une couche matérielle de densité uniforme. Une houle progressive vient de la partie libre où elle est sinusoïdale à grande distance. Comment cette houle est-elle modifiée dans la partie recouverte?"

Le problème revient à la détermination dans le demi-plan $y\leqq 0$ d'un plan complexe $z=x+iy$ d'une fonction harmonique ϕ définie par certaines conditions aux limites. Celles-ci après linéarisation s'écrivent: 1) $\phi_y-\phi=0$ pour $y=0$, $x>0$; 2) $\phi_y+c\phi=0$ pour $y=0$, $x<0$; 3) $\phi\to\cos(x+\alpha)$ ou $\sin(x+\alpha)$ pour $x\to+\infty$ avec $c=\delta g/(\delta_1\omega^2-\delta g)$, où g est l'accélération de la pesanteur, δ la densité de l'eau, δ_1 la densité superficielle de la couche recouvrante, ω la pulsation de la houle.

L'auteur traite d'ailleurs un problème un peu plus général. C'est celui où le demi-plan $y<0$ est remplacé par un secteur d'angle $\gamma<\pi$ et où (2) est remplacée par:

(2′) $\qquad \phi_n+c\phi=0 \quad \text{sur} \quad z=\rho e^{i\gamma},$

n désignant la dérivée normale extérieure. Si $c=0$, $\gamma<\frac{1}{2}\pi$ on a le problème posé par l'étude des vagues sur un fond incliné. Si $c=0$, $\gamma=\pi$, c'est le problème du dock. Ces cas particuliers ont été traités récemment par Stoker [Quart. Appl. Math. 5, 1–54 (1947); ces Rev. 9, 163], Lewy [Bull. Amer. Math. Soc. 52, 737–775 (1946); ces Rev. 9, 163], Isaacson [Comm. Pure Appl. Math. 3, 11–31 (1950); ces Rev. 12, 137], et Friedrichs et Lewy [ibid. 1, 135–148 (1948); ces Rev. 10, 336].

La solution du problème est donnée par une fonction $f(z)$, analytique dans le secteur et astreinte à vérifier une condition différentielle sur la frontière. L'auteur prend $f(z)$ sous la forme d'une intégrale prise sur un chemin indéterminé et sous le signe de laquelle figure une fonction $g(z)$ aussi indéterminée et il parvient à choisir convenablement cette fonction et le parcours d'intégration pour que $f(z)$ soit solution du problème. L'étude du comportement de cette fonction sur $y=0$, $x<0$, donne l'allure du mouvement à la surface dans sa partie recouverte. On obtient les résultats suivants: L'effet de la couche recouvrante dépend du signe de $c=\delta g/(\delta_1\omega^2-\delta g)$. Si $c>0$ la houle se propage avec une hauteur qui décroît comme $1/d$ pour d assez grand, d étant la distance au bord de la couche. Si $c<0$ la longueur de la houle est modifiée. Si $|c|$ est petit la houle dans la partie recouverte est longue et a une faible amplitude; si $|c|$ est grand elle est courte avec toujours une faible amplitude.

R. Gerber (Grenoble).

Roseau, Maurice. Sur les mouvements ondulatoires de la mer sur une plage. C. R. Acad. Sci. Paris 232, 211–213 (1951).

The Oy-axis is vertical, the plane Oxz is the horizontal undisturbed free surface, and $y+x\tan\alpha$ ($x>0$) is the equation of the beach or rigid bottom surface. In this, and in the three following reviews, the author is concerned with small irrotational motions of an incompressible fluid under gravity that lies in the infinite wedge between the free surface and the rigid bottom described above. Harmonic functions $\Phi(x, y, z; t)=e^{i\sigma t}\phi(x, y, z)$ must be found which are regular in the wedge, satisfy the conditions $\Phi_n=0$ at the bottom and $g\Phi_y+\Phi_{tt}=0$ on the free surface, are bounded at infinity, and have at most a logarithmic singularity at the origin. With the aid of such standing wave solutions, it is possible to construct progressing wave solutions [cf. the reviewer, Quart. Appl. Math. 5, 1–54 (1947); these Rev. 9, 163].

In the present paper the author gives, without details, the solution of the two-dimensional problem in the form of an integral representation for the complex velocity potential $f(z)$ whose real part is $\phi(x, y)$. The author makes use of a direct method, in contrast to E. Isaacson [Comm. Pure Appl. Math. 3, 11–31 (1950); these Rev. 12, 137] who gives the same result by analyzing the solutions of H. Lewy [Bull. Amer. Math. Soc. 52, 737–775 (1946); these Rev. 9, 163] for the special beach angles $(p/2q)\pi$ (p, q integers). The same problem has also been solved by A. S. Peters [see the preceding review] by a direct method. Peters also solves the more general problem in which the bottom condition $\phi_n=0$ is replaced by a mixed boundary condition $\phi_n+c\phi=0$ (c arbitrary).

J. J. Stoker, Jr. (New York, N. Y.).

Roseau, Maurice. Sur les mouvements ondulatoires de la mer sur une plage. C. R. Acad. Sci. Paris 232, 303–306 (1951).

In this paper the author shows that his solutions for the two-dimensional sloping beach problem valid for all angles of slope are the same as those given by H. Lewy and the reviewer for the special beach slopes $\alpha=\pi/2n$ (n an integer) [cf. the preceding review]. The author also gives the solution of a three-dimensional problem for the case $\alpha=\pi/4$ assuming the solution to be simple harmonic in the z-coordinate so that $\Phi=e^{i(kz+\sigma t)}\phi(x, y)$. The function $\phi(x, y)$ is thus a solution of a wave equation in two dimensions. The author's solution is obtained by the same method as was used by the reviewer for the case $\alpha=\pi/2$ [cf. the reference in the preceding review].

J. J. Stoker, Jr.

Roseau, Maurice. Sur les mouvements ondulatoires de la mer sur une plage. C. R. Acad. Sci. Paris **232**, 479–481 (1951).

A solution is given for the three-dimensional problem of waves over beaches sloping at any angle in case the waves are periodic in the z-coordinate [cf. the two preceding reviews].

J. J. Stoker, Jr. (New York, N. Y.).

Roseau, Maurice. Les mouvements ondulatoires de la mer sur une plage. C. R. Acad. Sci. Paris **231**, 1212–1214 (1950).

The sloping beach problem for arbitrary slope angles is formulated for both the two- and three-dimensional cases discussed in the three preceding reviews, but this time by means of integral equations which are derived by making use of appropriate Green's functions for Laplace's equation and the wave equation for an infinite plane sector. The integral equations are solved for the special case of the slope angle $\alpha=\pi/2$ [cf. the second preceding review].

J. J. Stoker, Jr. (New York, N. Y.).

Takahashi, Ryûtarô. On seismic sea waves caused by deformations of the sea bottom. Bull. Earthquake Res. Inst. Tokyo **20**, 375–400 (1942). (Japanese. English summary)

"Tunami, or seismic sea waves, are largely the result of deformations of the sea bottom, which in turn are caused by great earthquakes. This paper is a theoretical study of such sea waves. The sea is supposed to have uniform depth H, and the sea water to be incompressible. A circular portion of radius a of the sea bottom was assumed to have depressed during a time interval T by the amount $V/\pi a^2$, so that the total volume of the depression is V. The velocity of the depression was assumed uniform during interval T."

From the author's summary.

Takahasi, Ryûtarô. On seismic sea waves caused by deformations of the sea bottom. II. Bull. Earthquake Res. Inst. Tokyo **23**, 23–35 (1945). (Japanese. English summary)

"In a previous paper [see the preceding review], the writer has treated the seismic sea waves which would be generated when a circular portion, $2a$ in diameter, of the bottom of a sea having a uniform depth H, has depressed by a uniform velocity η in a time-interval T. In the present paper, the depression was assumed to have azimuthal and radial variations proportional to $r^n \cos n\theta$, other factors being the same as those in the previous calculation. The shapes of the deformation of the sea bottom are shown: $n=1$ gives an inclinational, $n=2$ a saddle-shaped, and $n=3$ a tripod-shaped deformation. The depression ζ of the sea surface at a distance r from the centre of the deformed area is also given." *From the author's summary.*

Takahasi, Ryûtarô. On seismic sea waves caused by deformations of the sea bottom. III. The one-dimensional source. Bull. Earthquake Res. Inst. Tokyo **25**, 5–8 (1947).

Takahasi, R. Transmission and reflexion of tunami waves at sea ridges and continental shelves. Bull. Earthquake Res. Inst. Tokyo **21**, 327–335 (1943). (Japanese)

Pocinki, Leon S. The application of conformal transformations to ocean wave refraction problems. Trans. Amer. Geophys. Union **31**, 856–866 (1950).

It is pointed out that rays in two-dimensional geometrical optics remain rays under conformal transformation of the plane. Thus by solving for the rays in a simple case in which the velocity depends upon one Cartesian coordinate only, and making conformal transformations of coordinates, the rays corresponding to the transformed velocity distribution are obtained. The terminology of water waves is used, rays being called orthogonals, since geometrical optics approximately describes the propagation of periodic surface waves of small amplitude. The examples treated are a circular island, a reef and a bay with parabolic bottom contours, and twin islands. The ratio of distances between neighboring rays is also computed since this quantity is necessary for the calculation of wave amplitudes. *J. B. Keller.*

Sekerž-Zen'kovič, Ya. I. On the theory of standing waves of finite amplitude on the surface of a heavy liquid of finite depth. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **15**, 57–73 (1951). (Russian)

In an earlier paper [Doklady Akad. Nauk SSSR (N.S.) **58**, 551–553 (1947); these Rev. **10**, 646] the author considered the problem of the title for water of infinite depth. The same method is applied in this paper for water of finite depth, and similar results are obtained. Lagrangian coordinates are used and the quantities of interest are expanded in a power series in a parameter ϵ, where $\epsilon=0$ corresponds to the standing waves obtained by the linearized theory. The computations for the first three coefficients in the expansions are carried through in detail and the corresponding form of the free surface is given. For proof of the convergence of the series and of the uniqueness of the solution reference is made to the method of the earlier paper. However, this paper was a brief note without proofs. There is no reference to recent work of R. Miche [Ann. Ponts Chaussées **114**, 25–78 (1944); these Rev. **7**, 348]. *J. V. Wehausen.*

Ursell, F. Trapping modes in the theory of surface waves. Proc. Cambridge Philos. Soc. **47**, 347–358 (1951).

The author shows by two examples that, even though the free surface extends to infinity in some direction, there can be modes of oscillation in the linearized theory of surface waves which contain finite total energy and for which the energy is not eventually transferred to infinity (hence, trapping modes). Consequently, the radiation condition at infinity is not sufficient to insure uniqueness. The two examples are (1) a sloping beach in a semi-infinite canal and (2) a submerged cylinder in an infinite canal.

J. V. Wehausen (Providence, R. I.).

Truckenbrodt, E. Ergänzungen zu F. Riegels: Das Umströmungsproblem bei inkompressiblen Potentialströmungen. Ing.-Arch. **18**, 324–328 (1950).

In two earlier papers [Ing.-Arch. **16**, 373–376 (1948); **17**, 94–106 (1949); these Rev. **10**, 490; **11**, 274] Riegels proposed a method to improve the familiar theory of thin airfoils in plane incompressible flow. His approximation consists essentially in a correction for finite thickness which would be exact in the case of an elliptic cylinder. To complete this work, the present author transforms Riegels' formulas for aerodynamic coefficients into sums over the profile ordinates with constant coefficients, tabulated here. In Riegels' second paper, the equivalent coefficients were calculated in terms of Fourier coefficients, which in turn were calculated from the ordinates. *W. R. Sears.*

Chien, Wei-Zang. The true leaving angle for diaphragm and bucket wheel with curved guides at the discharge end. Eng. Rep. Nat. Tsing Hua Univ. **4**, no. 1, 78–102 (1948).

An approximate method is given for the calculation of the discharge flow of a two-dimensional straight lattice in an incompressible flow utilizing the g-function of E. W. Barnes [Quart. J. Pure Appl. Math. **37**, 289–313 (1906)].

C. Saltzer (Cleveland, Ohio).

Toraldo di Francia, Giuliano. Moti uniformi di un liquido viscoso fra due sfere concentriche rotanti. Boll. Un. Mat. Ital. (3) **5**, 273–281 (1950).

The author considers the motion of a viscous incompressible fluid contained between two concentric spheres when the inner one rotates with a constant angular velocity ω and the outer one is fixed. Making use of a successive approximations method, he carries through to the third approximation the computation of the torque on the outer sphere, this being given as a function of R_1/R_2 and a Reynolds number $\omega R_2^2/\nu$, where R_1, resp. R_2, are radii of the inner, resp. outer, sphere and ν is the viscosity. Previous approximations had not been carried so far and had shown disagreement with experimental results.

J. V. Wehausen (Providence, R. I.).

Sowerby, L. The unsteady flow of viscous incompressible fluid inside an infinite channel. Philos. Mag. (7) **42**, 176–187 (1951).

"The problem considered is the motion of viscous incompressible fluid inside an infinite wedge-shaped channel of angle π/n, the channel being started suddenly from rest with uniform velocity in the direction of its length. The solution is obtained by considering a suitable Green's function for the heat conduction equation. The velocity is found to be expressible either in terms of error functions or in the form of integrals which are suitable for computation, depending on the value of n. In all cases the skin friction appears as a simple expression involving well known functions. Particular reference is made to the right angled channel ($n=2$), the results being extended, by Rayleigh's hypothesis [i.e., substituting z/w for t], to include some discussion of the steady flow inside a right angled channel having leading edges normal to the incident stream. From this, also, an approximation is derived for the frictional force on the interior of a finite rectangular channel of suitable dimensions."

The case $n=1$ is, of course, well known [cf. Lamb, Hydrodynamics, 6th ed., Cambridge Univ. Press, 1932, pp. 590, 591]. It is shown that $\tau_n=\tau_1$ as $r/2(\nu t)^{\frac{1}{2}}\to\infty$ where τ_n is the skin-friction for the case π/n and r is the distance from the axis. Also, when $n>1$, $\tau_n<\tau_1$.

J. V. Wehausen.

Schlichting, H. Einige exakte Lösungen für die Temperaturverteilung in einer laminaren Strömung. Z. Angew. Math. Mech. **31**, 78–83 (1951). (German. English, French, and Russian summaries)

In the conventional analysis of laminar flow the effect of viscous energy dissipation on the temperature distribution is neglected. Including this effect, the actual distributions are obtained for the three cases of laminar shear between a stationary and moving plane wall, pressure channel flow and flow between concentric cylinders. No effect of density variation is considered.

N. A. Hall.

Howarth, L. The boundary layer in three dimensional flow. I. Derivation of the equations for flow along a general curved surface. Philos. Mag. (7) **42**, 239–243 (1951).

The equations of boundary-layer flow along a curved surface S are set up in the system of orthogonal curvilinear coordinates ξ, η, ζ in which $\zeta=$const. gives S and all parallel surfaces, while $\xi=$const. and $\eta=$const. are the mutually orthogonal surfaces generated by the normals along the lines of curvature of S. In the resulting equations, the curvatures of the coordinate lines in S appear. It is shown that these equations are unique in spite of obvious arbitrariness in choosing the orthogonal coordinates. The curvature terms disappear only when S is a cylinder; then Cartesian coordinates can be used, as is customarily done in plane flow and for flow past yawed cylinders. If S is a surface of revolution, one of the two curvatures is zero and the other has a simple expression. In part II the author will discuss flow near a stagnation point on a general surface.

W. R. Sears.

Mager, Artur. Generalization of boundary-layer momentum-integral equations to three-dimensional flows including those of rotating system. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2310**, 47 pp. (1951).

Here the Navier-Stokes equations and the corresponding (Reynolds) equations with turbulent-stress terms are written for an orthogonal curvilinear coordinate system (x, y, z) that rotates about an arbitrarily chosen axis. The limitation is made that the coordinate system has curvature only in one plane, the plane of x and z. The boundary-layer approximations are then introduced to give the equations appropriate to incompressible laminar and turbulent boundary-layer flow along a smooth rotating surface. The author's assumption that (sufficiently small) curvature out of the (x, z)-plane would not affect the boundary-layer equations seems to be confirmed by Howarth [see the preceding review]. Next, the coordinate x is taken along streamlines of the potential flow and the boundary-layer equations are integrated with respect to y to give the momentum-integral equations for this case. The reduction of these to subcases considered by other investigators is shown.

The second part of this paper consists of approximate solution of the turbulent-boundary-layer momentum equations. Following Gruschwitz [Ing.-Arch. **6**, 355–365 (1935)] the profiles of the velocity components u and w (in the x and z directions) are assumed to be proportional to $(y/\delta)^n$ and $(y/\delta)^n(1-y/\delta)^2$, respectively, where δ is the boundary-layer thickness. These are compared with Gruschwitz's experimental profiles, measured in a nonrotating, curved duct. Using a skin-friction law of von Kármán, an approximate calculation is carried out for the boundary-layer growth along four streamlines of Gruschwitz's channel. Fair agreement is found, but the author is careful not to interpret this as a conclusive check of his method. The bibliography gives an apparently complete listing of papers on three-dimensional boundary layers published through 1949.

W. R. Sears (Ithaca, N. Y.).

Timman, R. A calculation method for three-dimensional laminar boundary layers. I. General theory. Nationaal Luchtvaartlaboratorium, Amsterdam. Report F. 66, i+24 pp. (1950).

A set of curvilinear coordinates is introduced, formed by a set of Gaussian coordinates on the body surface and the Euclidean distance from the body. It is assumed that the body surface is regular and either convex or moderately

concave. In particular, the Gaussian coordinates are specified to be equipotentials and streamlines of the potential flow at the surface. The form assumed by the steady, incompressible, laminar boundary-layer equations is attributed to Lin. These are integrated to give the momentum-integral relations for the boundary-layer flow. [For a comparable investigation, see the preceding review. Because of the difference in choice of coordinates it is not immediately evident that the momentum-integral equations of these two authors are the same in the present case.]

The author proposes to write the boundary-layer velocity components u_1, in the direction of the surface potential streamlines, and u_2, in the direction of the equipotentials, as being linearly composed from three basic functions of η, a dimensionless normal coordinate. This succeeds in reducing the momentum equations to a set of partial differential equations in two unknown functions. He then chooses these basic functions in a manner similar to his choice for two-dimensional boundary layers [same report series **F.35**, F29–F45 (1949); these Rev. **11**, 477]; namely, they are exponential functions and integrals thereof, having the asymptotic behavior at large η required by von Kármán and Millikan [Tech. Rep. Nat. Adv. Comm. Aeronaut. no. **504** (1934)]. In this approximation the various integrals required in the reduced momentum equations are evaluated. It is proposed to solve these equations numerically by the method of characteristics. Initial conditions are found from consideration of the stagnation point of the potential flow.

W. R. Sears (Ithaca, N. Y.).

Cooke, J. C. Pohlhausen's method for three-dimensional laminar boundary layers. Aeronaut. Quart. **3**, 51–60 (1951).

The same author recently [Proc. Cambridge Philos. Soc. **46**, 645–648 (1950); these Rev. **12**, 298] calculated the axial laminar boundary-layer flow for yawed infinite cylinders whose potential flow is given by $U=cx^m$, m having various constant values. Here he tests the Kármán-Pohlhausen method, as extended to cylindrical yawed flow by Wild [J. Aeronaut. Sci. **16**, 41–45 (1949)], by applying it to these cases. He uses two alternative procedures: in the first, called "full-Pohlhausen," both the chordwise (u) and axial (v) components are computed in Pohlhausen's (or Wild's) approximation; in the second, called "semi-Pohlhausen," the exact solutions $u(x,z)$ are used in the momentum-integral equation to find v. The agreement of both these methods with the exact (Cooke) results is good except for the most strongly retarded flow ($\beta=2m/(m+1)=-0.1988$); here the semi-Pohlhausen results are better, but still not extremely close. The author's conclusion is that Wild's method may not be worth the considerable labor of extending it to more general cases than the cylindrical, especially if retarded flow is involved. The paper closes with brief remarks about the calculation of swirl problems by Pohlhausen's procedure. Actually, the kind of extension of Wild's work that the author contemplates seems to have been undertaken recently by Timman [see the preceding review] although he does not use Pohlhausen's quartic functions.

W. R. Sears (Ithaca, N. Y.).

Hayes, Wallace D. The three-dimensional boundary layer. U. S. Naval Ordnance Test Station, Inyokern, Calif. Tech. Memo. **RRB-105**, ii+46 pp. (1950).

For three-dimensional laminar boundary layer flows of compressible and viscous fluids over smooth boundary surfaces, the equations of motion in curvilinear coordinates of the following cases are given: (a) the body of revolution; (b) cylindrical flow; (c) conical flow; and (d) streamline coordinates. It is shown that, except (b), all the cases considered cannot be generally reduced to two-dimensional problems but do admit solutions by an iteration method when the cross-flow is small. For case (d), the method of linearization of the cross-flow equation is briefly discussed. Finally, the integral forms of the equations are also presented.

Y. H. Kuo (Ithaca, N. Y.).

Moore, Franklin K. Three-dimensional compressible laminar boundary-layer flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2279**, 38 pp. (1951).

This author considers three-dimensional, compressible, laminar, boundary-layer flow over moderately curved surfaces. The "implicit" coordinate system introduced involves three distances, x, y, and $r(x)s$, where $r(x)$ has the dimension of length. It is particularly appropriate, for example, for bodies whose cross-sections are similar for various values of a distance coordinate x, so that $r(x)$ is a scale factor. The boundary layer equations are written in these terms, assuming that $y=0$ denotes the body surface. The energy equation is simplified by assuming constant specific heats and unit Prandtl number. Considering, for the moment, rectangular Cartesian coordinates, the author introduces functions ψ and φ such that (1) $\rho u=\psi_y$, $\rho v=-\psi_x-\varphi_z$, $\rho w=\varphi_y$, u, v, w being the corresponding velocity components, and ρ the density. These cause the equation of continuity to be satisfied identically. The existence, in general, of such functions is argued by considering the vector potential of mass flow. Next ψ and φ are redefined for the implicit coordinates and the boundary-layer equations are put in terms of $\psi(x,y,s)$ and $\varphi(x,y,s)$. Boundary conditions on φ and ψ are formulated.

Following Howarth [Proc. Roy. Soc. London. Ser. A. **194**, 16–42 (1948); these Rev. **10**, 270], the author assumes a linear relation between viscosity and temperature, and transforms ψ, φ, and y so as to bring the boundary-layer equations into a form resembling those for incompressible flow. A further transformation due to Mangler [Ministry of Aircraft Production [London], R. T. P. Translation no. **55**, GDC/689T (1946); Z. Angew. Math. Mech. **28**, 97–103 (1948); these Rev. **9**, 632] further simplifies the equations, reducing them to a form resembling familiar Cartesian cases. The reduction to known forms for plane, axisymmetric, and cylindrical flows is shown, and application is also made to a flat plate with arbitrary leading-edge contour. For certain contours, such as a pointed leading edge, there must be a "wake" extending back from the point, wherein the usual boundary-layer assumptions are invalid. The report closes with a discussion of the application to supersonic conical potential flows. It is suggested that the boundary-layer development along conical rays must be parabolic; the equations are modified under this assumption, but solutions are left for further investigations.

W. R. Sears.

***Kuerti, G. The laminar boundary layer in compressible flow.** Advances in Applied Mechanics, vol. 2, edited by Richard von Mises and Theodore von Kármán, pp. 21–92. Academic Press. Inc., New York, N. Y., 1951. $6.50.

This report presents a survey of methods and results concerning the steady laminar boundary layer in compressible flow. Only the two-dimensional problem is discussed. The purpose is to give "a guide to rather than a review of the existing literature on the subject." The main part of

the survey is "therefore, an organized representation of the various formal approaches that have evolved; it is written with the intent to show the common trait." After a brief but clear formulation of the basic equations, the principal characteristics of the problem are illustrated by the results obtained in the earlier works of Busemann, Crocco, and von Kármán on the boundary layer over a flat plate with Prandtl number $\sigma=1$. Following this, a large section called "The mathematics of boundary layer theory" gives a somewhat detailed account of all the published transformations and approximations of the basic equations. The exposition here is always lucid, and the motivation and the advantage of the different approaches are sometimes pointed out. The last part of this report is devoted to an exhibition of the main results obtained by the various authors, particularly for the flat plate problem. *H. Tsien* (Pasadena, Calif.).

***Lagerstrom, Paco A., Cole, Julian D., and Trilling, Leon. Problems in the Theory of Viscous Compressible Fluids.** California Institute of Technology, Pasadena, California, 1949. ii+iv+232 pp.

C'est le compte rendu des résultats obtenus dans la première partie de l'étude théorique entreprise à l'Institut de Technologie de Californie (Laboratoire Guggenheim) sur les effets de la viscosité dans les fluides compressibles. Quand la compressibilité ne peut pas être négligée, en particulier pour les écoulements supersoniques, la théorie classique de Prandtl sur la couche limite n'est plus suffisante pour représenter les phénomènes. Il devient alors nécessaire de considérer simultanément en une même région du fluide en mouvement la viscosité et la compressibilité. Mais en raison de la complexité mathématique des équations de Navier-Stokes pour de tels fluides il est nécessaire de faire des simplifications sur les équations pour pouvoir en étudier explicitement certaines solutions.

Au chapitre I est exposée l'obtention des équations du mouvement dans le cas général, puis celle des équations simplifiées par le linéarisation et par l'hypothèse que le coefficient de conductibilité thermique est nul. Il est vérifié que certaines propriétés caractéristiques du mouvement ne sont pas affectées, du moins qualitativement, par la linéarisation. La notion importante d'onde transversale et d'onde longitudinale est introduite. Il est montré qu'une onde satisfaisant aux équations linéarisées peut être décomposée en la somme d'une onde longitudinale et d'une onde transversale.

Le chapitre II traite principalement de la propagation des ondes. L'onde longitudinale à une dimension est étudiée dans le détail (c'est le mouvement supposé par tranches planes engendré par un piston). Cet exemple montre le rôle de la viscosité: les caractèristiques ne reprisentent plus la propagation d'une discontinuité, mais d'une variation des vitesses, qui si elle peut être rapide n'en est pas moins continue. Le problème des ondes transversales monodimensionnelles est ensuite analysé. Elles sont crées par un plan se déplaçant sur lui-même. La solution est indépendante de la compressibilité et les caractéristiques ne jouent plus de rôle. En ce qui concerne les conditions à la frontière la viscosité est de peu d'importance pour les ondes longitudinales tandis qu'elle a un rôle primordial pour les ondes transversales.

La majeure partie du rapport est consacrée à l'étude des problèmes linéarisés. Il était pourtant intéressant de voir, au moins qualitativement ce que pouvaient introduire les termes non linéaires. C'est ce qu'on fait les auteurs, en se limitant à des exemples simples du fait de l'extrême complexité du problème. Dans un appendice cet effet a été étudié théoriquement pour l'onde longitudinale à une dimension. Dans un autre appendice le cas de l'onde transversale monodimensionnelle est examiné par un procédé d'itération: une onde de pression longitudinale existe alors ce qui ne se produisait pas pour le problème linéarisé. Dans la fin du chapitre les ondes à plusieurs dimensions sont introduites. Ce sont en particulier les ondes longitudinales engendrées par une sphère en pulsations et les ondes transversales dues à un cylindre en rotation. L'étude du mouvement créé par un plan infiniment petit soumis à une accélération infinie est d'un grand intêret théorique. Dans ce cas schématique les ondes ont une composante transversale et une composante longitudinale. La première est étudiée à partir du champ du tourbillon qui est celui d'un doublet.

Dans le dernier chapitre sont étudiés des problèmes aux limites. Un problème qui se pose est celui de l'étude du mouvement engendré par une portion de plan qui se déplace parallèlement à elle-même. L'analyse utilise les résultats acquis sur les ondes. Il est établi en particulier que dans le cas supersonique des ondes longitudinales prennent naissance sur le bord du plan. Une importante partie du chapitre est consacrée aux solutions fondamentales. Ce sont elles qui donnent le champ des vitesses pour une impulsion tangentielle appliquée en un point du fluide (ce qui correspond à un morceau de plan infiniment petit animé d'une vitesse tangentielle infinie). *R. Gerber* (Grenoble).

Obuhov, A. M., and Yaglom, A. M. The microstructure of a turbulent flow. Akad. Nauk SSSR. Prikl. Mat. Meh. 15, 3–26 (1951). (Russian)

This is a joint exposition of work on locally isotropic turbulence by the authors which has been published in the last few years in separate notes. Let M and M' be two points at a distance r in a turbulent fluid, $v_l(M)$ and $w_l(M)$ the components of the velocity and acceleration, respectively, in the direction MM', $p(M)$ the pressure, and

$$D_{ll}(r)=\overline{[v_l(M')-v_l(M)]^2},\quad \Pi(r)=\overline{[p(M')-p(M)]^2},$$
$$A_{ll}(r)=\overline{w_l(M)w_l(M')}.$$

In the first part of this paper the known fundamental equations of locally isotropic turbulence are derived and then the behavior of $D_{ll}(r)$ computed under the assumption that the skewness $S=D_{lll}(r)/[D_{ll}(r)]^{\frac{3}{2}}$ is a constant ($=-0.4$) for the whole range of r [cf. Obuhov, Doklady Akad. Nauk SSSR 67, 643–646 (1949); these Rev. 11, 66]. [Reviewer's note: The most recent measurements of S by R. W. Stewart [Proc. Cambridge Philos. Soc. 47, 146–157 (1951); especially pp. 156, 157] indicate that S is not constant in either r or the Reynolds number.]

In part two the behavior of $\Pi(r)$ is computed making use of the above assumption on S and an additional assumption that the fourth moments of the velocity field behave as in a Gaussian distribution [cf. Obuhov, ibid. 66, 17–20 (1949); these Rev. 10, 757]. In the last part the accelerations and the pressure gradients are considered [cf. Yaglom, ibid. 67, 795–798 (1949); these Rev. 11, 280] and A_{ll} is computed assuming S constant. In the first and last parts the corresponding quantities for the components normal to MM' are also considered (so that the corresponding tensors may be constructed). The computed results are presented graphically in suitable nondimensional form. The asymptotic values for $r\rightarrow 0$ and $r\rightarrow\infty$ are also considered in each case. [See also the following review.] *J. V. Wehausen.*

Batchelor, G. K. Pressure fluctuations in isotropic turbulence. Proc. Cambridge Philos. Soc. **47**, 359–374 (1951).

This paper treats approximately the same problems considered in parts 2 and 3 of the paper reviewed above. However, the author is more concerned with the function $P(r)=\overline{p(M')p(M)}/\rho^2$ than with the function $\Pi(r)$ of the preceding review. Also, he assumes homogeneous isotropic turbulence. The same assumption on the behavior of the fourth moments is made, but not the assumption on S. In addition, the spectral functions corresponding to the various correlation functions are derived. The results of the two papers seem to be in agreement where they overlap (except for choice of constants). In particular, the author derives the relations $P(r)=2(\overline{u^2})^2\int_r^\infty(y-r^2y^{-1})f'^2dy$, $\overline{p^2}/\rho^2=P(0)$, $\overline{(\text{grad }p)^2}=-3P''(0)$. Here $f(y)$ is the usual longitudinal correlation function of isotropic turbulence and is related to the function $D_{11}(r)$ of the preceding review. For the case of very large Reynolds numbers it is found that $\overline{p^2}=0.34\rho^2(\overline{u^2})^2$. Assumption of a special form for the function $D_{11}(r)$ leads to $\overline{(\text{grad }p)^2}=3.9\rho^2\nu^{-\frac{1}{2}}\epsilon^{\frac{3}{2}}$ for large Reynolds numbers. *J. V. Wehausen* (Providence, R. I.).

Limber, D. Nelson. Numerical results for pressure-velocity correlations in homogeneous isotropic turbulence. Proc. Nat. Acad. Sci. U. S. A. **37**, 230–233 (1951).

Using the same assumptions as in the paper reviewed above, the author expresses $\overline{p(M)u_i(M')u_j(M')}$ as an integral similar to the one for $P(r)$ in the preceding review. For the case of infinite Reynolds number the results are given numerically. *J. V. Wehausen* (Providence, R. I.).

Lucke, O. Bemerkungen zur Definition des Austauschtensors. I. Z. Meteorologie **4**, 216–222 (1950).

Akita, Yosio. Non-linear character of the compressible aerodynamics. Jap. Sci. Rev. Ser. I. **1**, no. 2, 5–10 (1950).

Solutions to the nonlinear partial differential equation for the velocity potential φ in plane, irrotational flow of a polytropic gas are sought for as power series in r and r^{-1}, where $r=(x^2+y^2)^{\frac{1}{2}}$, the coefficients being unknown functions of the polar angle θ. If M denotes the Mach number in the free stream, the failure of this expansion for critical Mach numbers is used to infer that shock waves must appear. By use of variational methods coupled with the method of Ritz the author finds that the lift coefficient for an airfoil nearly elliptical in shape takes its maximum value before the critical Mach number is reached. *M. H. Martin.*

Martin, M. H. Steady, plane, rotational Prandtl-Meyer flow of a polytropic gas. J. Math. Physics **29**, 263–281 (1951).

The author studies isoenergetic, anisentropic, Prandtl-Meyer flows of a polytropic gas ($p\propto\rho^\gamma$), where a Prandtl-Meyer flow is defined as a flow whose hodograph reduces to a single arc $\theta=\theta(q)$. It has been shown that such flows occur only for a special entropy distribution:

$$S(\psi)=(c_p/\lambda)\log(\psi-\psi_0)+\text{const.}\ (\lambda\neq0),$$

or $S(\psi)=A(\psi-\psi_0)$ (S=entropy, ψ=stream function) [M. H. Martin, same vol., 76–89 (1950); these Rev. **12**, 215]. First, the hodograph curve h: $\theta=\theta(q)$ is shown to satisfy

$$d\theta/dq=\pm q_0c^{-2}(q^2-c^2)[q^2(1-q^2)(q^2-q_0^2)+Kq^4(1-q^2)^{-2\lambda}]^{-\frac{1}{2}},$$

with $c^2=(\gamma-1)/(\gamma+1)$, $q_0=[(2\lambda+1)/(2\lambda c^2+1)]^{\frac{1}{2}}c^2$, K being an arbitrary constant. Detailed analysis then follows for the case $K=0$. Thus, h is shown to be a family of roulettes. Next, it is found that the isovels in the physical plane are a family of parallel or concurrent straight lines, and that each isovel meets the stream lines at a constant angle. Finally, the stream lines, isobars (p=const.), and isopycnics (ρ=const.) are all expressible in polar coordinates (R,ω) as $R=R_0(\sec c^2\omega/q_0)^e$, where $e=e_\psi=c^{-2}$, $e=e_p=-(2\lambda+1)$, $e=e_\rho=1-2\lambda/\gamma$ for the respective lines. According to the value of q_0, various flow patterns are obtained, such as corner flow ($0<q_0<c^2$), half-plane flow ($q_0=c^2$), flow around a wedge ($c^2<q_0\leqq2c^2$), and overlapping flow ($2c^2<q_0<1$). *I. Imai* (Tokyo).

Riabouchinsky, Dimitri. Sur les écoulements permanents subsonique, sonique et supersonique presque uniformes. C. R. Acad. Sci. Paris **232**, 280–283 (1951).

For the linearized velocity potential equation

$$(1-M^2)\partial^2\varphi/\partial x^2+\partial^2\varphi/\partial y^2+\partial^2\varphi/\partial z^2=0,\quad 0<M<\infty,$$

the author considers solutions $\varphi=x+\alpha f(x,y,z)$ (α^2 negligible) based on plane wave solutions

$$f=(M^2-1)^{-\frac{1}{2}}\sum[\mathfrak{F}_\nu(\zeta)-\mathfrak{F}_\nu(\bar\zeta)]+\sum[\mathfrak{F}_\mu(\zeta)+\mathfrak{F}_\mu(\bar\zeta)]$$

for arbitrary functions $\mathfrak{F}_\nu$, where

$$\zeta(\text{or }\bar\zeta)=x+(\text{or }-)(M^2-1)^{\frac{1}{2}}(a_{1\nu}y+a_{2\nu}z)$$

and $a^2_{1\nu}+a^2_{2\nu}=1$. Let

$$f_i=-\sum a_{i\nu}[\mathfrak{F}_\nu(\zeta)+\mathfrak{F}_\nu(\bar\zeta)]-(M^2-1)^{\frac{1}{2}}\sum a_{i\mu}[\mathfrak{F}_\mu(\zeta)-\mathfrak{F}_\mu(\bar\zeta)]$$

where $i=1,2$, and let $\psi_1=y+\alpha f_1$, $\psi_2=z+\alpha f_2$. To the first order in α, ψ_i=constant are stream sheets, and $x=\varphi-\alpha f(\varphi,\psi_1,\psi_2)$, $y=\psi_1-f_1(\varphi,\psi_1,\psi_2)$, $z=\psi_2-\alpha f_2(\varphi,\psi_1,\psi_2)$ is a parametric representation of the flow. As $M\to1$, $\frac{1}{2}f\to\sum(a_{1\nu}\psi_1+a_{2\nu}\psi_2)d\mathfrak{F}_\nu(\varphi)/d\varphi+\sum\mathfrak{F}_\mu(\varphi)$, etc. Such a "sonic" flow along a wavy cylindrical wall has been constructed by starting from $f=(\cos\zeta-\cos\bar\zeta)/(M^2-1)^{\frac{1}{2}}$. Considering the inadequacy of the linearized equations for description of transonic flow, the reviewer believes the passage to the limit $M=1$ to be of merely formal interest.
J. H. Giese (Havre de Grace, Md.).

Herriot, John G. Blockage corrections for three-dimensional-flow closed-throat wind tunnels, with consideration of the effect of compressibility. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **995**, 13 pp. (1950).

The constriction due to a test model placed in a closed-throat wind tunnel causes the flow past the model to speed up ("solid-blockage") and the flow in the wake to slow down ("wake-blockage"). The author summarises the existing literature, calls attention to certain errors, and presents formulae and tables for making blockage corrections to observed wind-speed, pressure, Mach number, and drag coefficient for a model in the form of a solid of revolution on a three-dimensional unsweptback wing centrally located in a tunnel whose cross-section is circular or rectangular.
L. M. Milne-Thomson (Greenwich).

Rott, Nikolaus. Flügelschwingungsformen in ebener kompressibler Potentialströmung. Z. Angew. Math. Physik **1**, 380–410 (1950).

"On the basis of energy considerations a survey is given of the possible forms of oscillations for flutter with two degrees of freedom in a plane compressible potential flow. It is found that the possible forms do not much depend on Mach number in the whole range from 0 to ∞, with some exceptions in the neighbourhood of $M=1$, where forms with

only one degree of freedom (pure torsion) may occur. Conclusions are drawn for methods of preventing flutter. Limits of the reduced frequency (depending on M) are given for the possibility of flutter in two and one degree of freedom. Special care was given to the case $M=1$, for which analytical expressions and numerical values of the derivatives are presented." (From the author's summary.)

The reviewer notes that, while it is true that linearised theory is valid in the transonic range for sufficiently large values of $(\text{thickness})^{-\frac{1}{3}}\cdot(\text{reduced frequency})$ [cf. Lin, Reissner, and Tsien, J. Math. Physics **27**, 220–231 (1948); these Rev. **10**, 162], only very moderate thickness ratios (circa 3–5%) render torsional damping positive for infinite aspect ratio, as also will sufficiently small, effective aspect ratio, even for zero thickness. *J. W. Miles* (Auckland).

Weinstein, Alexander. Transonic flow and generalized axially symmetric potential theory. Symposium on theoretical compressible flow, 28 June 1949. Naval Ordnance Laboratory, White Oak, Md., Rep. **NOLR-1132**, pp. 73–82 (1950).

A transonic flow may be described approximately by Tricomi's equation (1) $\sigma\psi_{\theta\theta}+\psi_{\sigma\sigma}=0$, where ψ is the stream-function and σ, θ appropriate hodograph variables. For $\sigma<0$ (1) may be written in characteristic coordinates as (2) $(\eta^k\psi_\xi)_\xi+(\eta^k\psi_\eta)_\eta=0$ (with $k=\frac{1}{3}$). The author studied equation (2) in a previous paper [Trans. Amer. Math. Soc. **63**, 342–354 (1948); these Rev. **9**, 584], interpreting ψ as a rotationally symmetrical solution of the Laplace equation in a $(k+2)$-space. The resulting formulas are valid for non-integral k. Thus the author obtains a fundamental solution of (1) in the form (3) $\psi_b(\xi,\eta)=\pi^{-1}b^k\eta^{-k/2}Q_{-k/2}(1+2\epsilon^2)$ where Q is a Legendre function of the second kind and $\epsilon^2=[\xi^2+(\eta-b)^2]/4b$. Other solutions of (1) with singularities are obtained from (3) by differentiation and integration. All these functions can be continued into the "subsonic" (elliptic) region $\sigma>0$ without singularities. As an application a formula is derived representing a subsonic flow past a wedge in a channel which becomes sonic along curves joining the edge-walls with the channel-walls. Other applications are to be given in subsequent papers. *L. Bers.*

Robinson, A., and Young, A. D. Note on the application of the linearised theory for compressible flow to transonic speeds. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2399 (10,474)**, 6 pp. (1951).

See Coll. Aeronaut. Cranfield. Rep. no. **2** (1947); these Rev. **9**, 477.

Kaplan, Carl. On a solution of the nonlinear differential equation for transonic flow past a wave-shaped wall. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2383**, 35 pp. (1951).

The perturbation of the potential function from the uniform flow value is proportional to a function f which satisfies the equation $f_{yy}=f_xf_{xx}$ in the transonic range in which terms of order higher than the first in $1-M^2$ are neglected. By analogy with the Prandtl-Busemann iterative solution, the author takes $f=-x+\sum_{n=1}^{\infty} f_n \sin(nx)$, where the f_n depend on y. Substituting into the differential equation he gets the equations

$$(*)\quad d^2f_n/dy^2-n^2f_n=-\tfrac{1}{4}n\sum_{m=0}^{n} m(n-m)f_mf_{n-m}$$
$$-\tfrac{1}{2}n\sum_{m=0}^{\infty}(m+1)(m+n+1)f_{m+1}f_{m+n+1},$$

for $n=1, 2, \cdots$. These are solved by expanding

$$(**)\qquad f_n=\sum_{p=0}^{\infty} e^{-(n+2p)y}P_{n,p}(y),$$

where the $P_{n,p}$ are polynomials, inserting (**) into (*) and equating coefficients of corresponding exponentials. This device has the effect of expressing these polynomials in terms of lower order ones which may then be computed one by one from the beginning. The difficult question of convergence is partially investigated. (**) definitely fails to converge if the wall wave amplitudes exceed a certain limit, but the $P_{n,p}(y)$ rapidly become small if these amplitudes are sufficiently small. *E. Pinney* (Berkeley, Calif.).

Robinson, A. Wave reflexion near a wall. Proc. Cambridge Philos. Soc. **47**, 528–544 (1951).

Originally issued as Coll. Aeronaut. Cranfield. Rep. no. **37** (1950); these Rev. **12**, 454.

Ferri, Antonio. Supersonic flow around circular cones at angles of attack. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2236**, 30 pp. (1950).

The supersonic flow around cones without axial symmetry is considered. It is shown that singular points must exist in any conical flow without axial symmetry. For flow around a circular cone at a small angle of attack, the singular points take place on the surface of the cone and in consequence there appears a thin vortical layer, in which pressure remains constant but entropy varies abruptly in the direction normal to the surface. Taking account of the existence of the vortical layer, a numerical method for the determination of the flow field around cones is developed on the assumption that the second order terms of angle of attack can be neglected. The method is applied to cones at finite angles of attack, and good agreement with experimental results is obtained.

I. Imai (Tokyo).

Jones, W. Prichard. Summary of formulae and notations used in two-dimensional derivative theory. Ministry of Aircraft Production, Aeronaut. Res. Committee, Rep. and Memoranda no. **1958 (5772)**, 21 pp. (1942).

Ernsthausen, Wilhelm. Der rotierende Tragflügel als Strahlungsproblem. Z. Angew. Math. Mech. **31**, 20–35 (1951). (German. English, French, and Russian summaries)

The author takes partial account of the effects of compressibility of the air on a rotating airfoil (airscrew) by regarding it as a generator of acoustical radiation. The motion of the generator is resolved into the components of a Fourier time series and these are treated individually by acoustical theory. Simple mechanical analogues are described. The simplifying assumption is made that the spatial distribution of acoustic source intensity is constant over the airfoil. In many important propeller vibration modes this is not the case, so the theory would have to be extended to be used in practical applications. The elastic properties of the propeller would also have to be treated.

E. Pinney (Berkeley, Calif.).

Westervelt, P. J. The theory of steady forces caused by sound waves. J. Acoust. Soc. Amer. **23**, 312–315 (1951).

The author's summary is as follows: "A general expression is derived for the force owing to radiation pressure acting on an object of any shape and having an arbitrary normal boundary impedance. It is shown that boundary layer

losses may lead to forces that are several orders of magnitude greater than the forces owing to classical radiation pressure. Steady forces arising from an asymmetric wave form are compared with the other forces. A sound wave, consisting of equal parts of fundamental and second harmonic components, can cause forces ten or more orders of magnitude greater than the forces owing to radiation pressure to be exerted in small particles." *Y. H. Kuo* (Ithaca, N. Y.).

Hart, Robert W. Sound scattering of a plane wave from a nonabsorbing sphere. J. Acoust. Soc. Amer. **23**, 323–329 (1951).

The author considers the scattering of sound by a spherical region whose acoustic properties, as given by its density and the velocity of sound, differ from that of the surrounding medium. The consequent boundary value problem may be readily solved in terms of an infinite series which is generally convenient only as long as the sphere radius is small compared to the wavelength. The author obtains an approximate closed expression for the angular distribution which turns out to be valid when the acoustic properties of the sphere are close to that of the surrounding medium. Further approximations are required to obtain the total cross-section. Comparison with exact calculations is made.
H. Feshbach (Cambridge, Mass.).

Ingard, Uno. On the reflection of a spherical sound wave from an infinite plane. J. Acoust. Soc. Amer. **23**, 329–335 (1951).

The scattering of a spherical sound wave by a plane barrier is discussed. The equation to be solved is $\nabla^2\psi+k^2\psi=0$. The boundary conditions (1) limit the singularities of ψ to one at the source of the spherical wave and (2) require $\partial\psi/\partial n=a\psi$ on the plane where a is a constant. It is shown that the reflected wave may be represented as a spherical wave arising from the image point with a source strength Q which is a function of a and of the angle between the line connecting the image and the observation point and the line between source and image; an approximate value for Q is obtained. Numerical calculations are made for several values of a as well as for one case where the answer is known. The comparison is very favorable. *H. Feshbach.*

Pachner, Jaroslav. On the acoustical radiation of an emitter vibrating in an infinite wall. J. Acoust. Soc. Amer. **23**, 185–198 (1951).

The author's aim is to provide useful formulae for the numerical calculation of the sound field in the plane of a circular emitter vibrating harmonically in an infinite rigid baffle. Substituting in Rayleigh's classical formula, viz.

$$\psi=-(1/2\pi)\int_0^\infty\int_0^{2\pi}(\partial\psi/\partial z)(e^{ikR}/R)\rho d\rho d\varphi,$$

a certain expansion for e^{ikR}/R which was suggested by L. V. King [Canadian J. Research Sect. A. 11, 135–155 (1934)] and the reviewer [Philips Research Rep. 1, 251–277 (1946)], the author obtains, for points in the plane of the emitter,

$$\psi(r,\varphi_0,0)=-\sum_{m=0}^\infty(\epsilon_m/2\pi)\int_0^\infty\int_0^{2\pi}\{\partial\psi(\rho,\varphi,0)/\partial z\}$$
$$\times\cos m(\varphi-\varphi_0)I_m(r,\rho)\rho d\rho d\varphi,$$

in which

$$I_m(r,\rho)=\int_0^\infty J_m(\lambda r)J_m(\lambda\rho)(\lambda^2-k^2)^{-\frac12}\lambda d\lambda.$$

The imaginary part of this Bessel-function integral is easy to evaluate, viz.

$$\Im[I_m(r,\rho)]=(1/r)\sum_{n=0}^\infty\binom{n-\frac12}{n}(\rho/r)^n$$
$$\times J_{m+n}(k\rho)(\pi kr/2)^{\frac12}J_{m+n+\frac12}(kr).$$

[The statement that this series converges only if $r>\rho$, is apparently incorrect. The author's series (9a) and (9b) converge for all real values of r and ρ. The corresponding series for $\Re[I_m(r,\rho)]$ behave differently in this respect.] The real part of $I_m(r,\rho)$ is more troublesome. Using various transformations of Bessel functions and hypergeometric functions, the author succeeds in deriving a simple result for the case $m=0$ only, viz.

$$I_0(r,\rho)=ik\sum_{n=0}^\infty\binom{n-\frac12}{n}(\rho/r)^nJ_n(k\rho)\zeta_n^{(1)}(kr),\qquad r>\rho,$$

and a similar expression if $r<\rho$. An alternative expansion is given without proof, viz.

$$I_0(r,\rho)=ik\sum_{n=0}^\infty(1+4n)\left[\binom{n-\frac12}{n}\right]^2(1/k\rho)(\pi k\rho/2)^{\frac12}$$
$$\times J_{2n+\frac12}(k\rho)\zeta_{2n}^{(1)}(kr),\quad r>\rho.$$

The author then applies his theory to the following type of velocity distribution on the disk (radius r_0):

$$-(\partial\psi/\partial z)_{z=0}=v_0(1-\rho^2/r_0^2)^q.$$

It is found that, if $r>r_0$,

$$\psi(r)=(iv_0r_0)(2/kr_0)^qq!\sum_{n=0}^\infty\binom{n-\frac12}{n}\left(\frac{r_0}{r}\right)^nJ_{n+q+1}(kr_0)\zeta_n^{(1)}(kr).$$

The corresponding expression for $r<r_0$ is too complicated to warrant its quotation; it is given explicitly for $q=0, 1, 2$. [Simpler expressions are obtained (but unpublished) by the reviewer.] The remainder of the paper deals with series expansions in spherical wave functions. The reviewer is unable to give a fair account of the author's highly abstract formulation in terms of Dirac's bra and ket vectors. Some useful expressions involving Bessel functions are given in the appendix. *C. J. Bouwkamp* (Eindhoven).

Pachner, Jaroslav. On the acoustical radiation of an emitter vibrating freely or in a wall of finite dimensions. J. Acoust. Soc. Amer. **23**, 198–208 (1951).

The radiation field of the emitter mentioned in the title is considered as a superposition of (1) the field of the same transmitter in an infinite baffle and (2) a field with vanishing normal derivative on the baffle and the emitter so as to make the total field vanish in the free-space part of the baffle-emitter plane. The author assumes the field (1) as known. Then part (2) is determined by one of Rayleigh's integral equations, which is solved by a double infinite series of eigenfunctions. The author's final results are abstractly written in terms of Dirac's bra and ket vectors [cf. the preceding review]. *C. J. Bouwkamp* (Eindhoven).

Hollmann, Günther. Eine Überführung des ersten Wärmehauptsatzes in die hydrodynamischen Gleichungen. Z. Meteorologie 4, 222–229 (1950).

The author deals with the usual meteorological equations in which the earth's surface is approximated as plane: i.e. vertical acceleration is neglected. The earth's atmosphere is taken as a frictionless perfect gas. The author's contribu-

tion is, taking the potential temperature $\Theta = T(p_n/p)^{(\gamma-1)/\gamma}$, where p_n is a reference pressure, as an independent variable, to eliminate the energy equation. The resulting system, as he points out, although valid without any particular assumption regarding the nature of energy exchange S is not essentially more complicated than the usual one in which the motion is assumed isentropic. Let α_x, α_y denote the absolute acceleration components for a plane motion, calculated at $\Theta = \text{const.}$; let Q denote the rate of heat addition per unit mass and time; let M be the sum of the specific enthalpy and the potential energy; then the author's final equations are

$$\alpha_x + \frac{\Theta}{c_pT}\frac{\partial v_x}{\partial \Theta}Q + \frac{\partial M}{\partial x} = 0, \quad \alpha_x + \frac{\Theta}{c_pT}\frac{\partial v_y}{\partial \Theta}Q + \frac{\partial M}{\partial y} = 0,$$

$$\frac{\partial}{\partial x}\left(v_x\frac{\partial p}{\partial \Theta}\right) + \frac{\partial}{\partial y}\left(v_y\frac{\partial p}{\partial \Theta}\right) + \frac{\partial}{\partial \Theta}\left(\frac{\Theta}{c_pT}\frac{\partial p}{\partial \Theta}Q\right) + \frac{\partial^2 p}{\partial \Theta \partial t} = 0.$$

He puts the vorticity theorem of Ertel [Meteorologische Z. **59**, 277–281 (1942)] into the following simple form (subject, of course, to his initial approximations): $W/(\partial p/\partial \Theta)$ is constant for each particle, where W is the vertical component of absolute vorticity evaluated at $\Theta = \text{const}$. He draws some qualitative meteorological conclusions from these results, which he uses also to put the equation of Bjerknes for the meridional circulation into a simpler form.

C. Truesdell (Bloomington, Ind.).

Perkins, D. T. On the solutions of the equations of motion for linear fields. J. Meteorol. **7**, 291–303 (1950).

Assuming that the pressure force has a potential which is quadratic in x, y, and t, the equations of horizontal frictionless motion are solved, neglecting the variation of the Coriolis parameter. The general solution may be split into two components, called the central and the eccentric, each of which satisfies the equations of a linear velocity field. The absolute vorticity of the geostrophic wind and the discriminant of the potential provide a means of classification of stream functions and determine the nature of the general solution. The central component comprises the low frequency oscillations and has positive absolute vorticity, while the eccentric component has relatively high frequency oscillations and negative absolute vorticity. The central field is found to be a stable approximation of the true wind, to be regarded as a generalization of the geostrophic wind field, taking into account the effects of contour curvature, contour convergence, geostrophic shear, and temporal changes of contour gradient ("contour" means "member of family of equipotential lines"). The eccentric field is found to be a horizontal oscillation with period a function of the discriminant of the potential and the absolute vorticity of the geostrophic wind. Methods for applying the results to actual weather charts are given, but statistical evaluation of the worth of the approximations has not yet been carried out. Since rectangular coordinates were used, results are applicable only in a small neighborhood of a point on the earth's surface. *W. D. Duthie* (Monterey, Calif.).

Rouaud, A. Quelques considérations sur les trajectoires et les lignes de courant. Cas d'un cyclone mobile. J. Sci. Météorologie **2**, 33–49 (1950). (French. English and Spanish summaries)

Trajectories with respect to a polar coordinate system with pole at the center of a circular cyclone moving with constant velocity are determined for two particular cases, one in which the (gradient) wind speed is everywhere constant, the other in which it is an exponential function of the radius vector, zero at the center, rising to a maximum some distance from the center, and then decreasing asymptotically to zero. Absolute trajectories are then determined from the relative trajectories. The deformation of frontal systems is studied in each of the two cases by determining the displacement of selected points on the fronts. The resultant deformations are interesting but somewhat unrealistic, since the wind discontinuity normally associated with fronts is ignored. *W. D. Duthie* (Monterey, Calif.).

Berlyand, O. S. The distribution of atmospheric pressure on the surface of the earth in the case of steady zonal circulation of the atmosphere. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **14**, 255–259 (1950). (Russian)

By assuming a coefficient of turbulent transfer of momentum which is linear in the vertical, the author shows that the equations of motion for steady zonal flow on a spherical earth reduce to a Bessel equation in which the specific momentum is the dependent variable. Because the motion is purely zonal, the resultant solution can be transformed into a simple integral expression for the (mean) meridional pressure gradient. Using observed mean temperature data, the mean meridional sea-level pressure profile is computed and compared with observed (Russian) data. Qualitative agreement is good. The computed subtropical pressure maximum is smaller than the observed value in magnitude and is displaced slightly poleward from the observed position. On the other hand, the computed profile shows a pressure minimum exactly at the pole and lower than the observed value, thus lending theoretical support to the claim of Russian meteorologists that no secondary sea-level pressure maxima exist at the poles. A pressure minimum at the equator was assumed as part of the boundary conditions; the computed polar minimum is lower than the equatorial minimum. *W. D. Duthie.*

van Mieghem, Jacques. Sur la circulation transversale associée à un courant atmosphérique. Tellus **2**, 52–55 (1950).

L'auteur critique l'introduction d'une loi causale dans l'interprétation du théorème de la circulation de V. Bjerknes. En calculant la circulation transversale d'un courant horizontal, l'auteur montre qu'elle peut être décomposée en une somme de cinq termes dont deux sont prépondérants; il étudie l'influence de chacun de ces termes sur le sens de la circulation. Malheureusement il est encore difficile dans l'état actuel de nos connaissances de préciser l'ordre de grandeur de ces termes. *M. Kiveliovitch* (Paris).

van Mieghem, Jacques. Sur le mouvement isobarique de l'air atmosphérique. Arch. Meteorol. Geophys. Bioklimatol. Ser. A. **2**, 65–72 (1950).

The Rossby formula for the speed of waves in a zonal current is derived for perturbations in an isobaric surface. The author was apparently unaware that this has already been done by A. Eliassen [Geofys. Publ. Norske Vid.-Akad. Oslo **17**, no. 3 (1949)]. *W. D. Duthie.*

Platzman, George W. The motion of barotropic disturbances in the upper troposphere. Tellus **1**, no. 3, 53–64 (1949).

L'auteur étudie les oscillations d'un modèle de fluide barotrope de la troposphère supérieure en introduisant une circulation assez intense des vents d'ouest. On ne considère

que les composantes horizontales du vent et on suppose que le mouvement perturbé est irrotational. Il s'agit de résoudre une équation de Laplace à deux dimensions, en coordonnées polaires. L'auteur détermine ensuite les fréquences fondamentales et étudie en détail le mouvement.
M. Kiveliovitch (Paris).

Munk, W. H., and Carrier, G. F. The wind-driven circulation in ocean basins of various shapes. Tellus 2, 158–167 (1950).

La question a déjà été étudiée par un des auteurs pour un bassin de forme rectangulaire [W. H. Munk, J. Meteorol. 7, 79–93 (1950)]. Dans ce mémoire les auteurs considèrent le cas d'un bassin de forme triangulaire. En introduisant une fonction ψ qui est le transport de masse de la fonction du courant, on montre que cette fonction satisfait à une équation aux dérivées partielles du 4ème ordre, équation obtenue déjà dans le mémoire cité. Comme on voit facilement, cette équation est une interprétation intégrale de l'équation du mouvement tourbillonnaire. En ramenant cette équation à une forme non dimensionnelle, on s'aperçoit que le coefficient des termes du 4ème ordre est très petit. Les auteurs cherchent une solution asymptotique et montrent que cette solution constitue une très bonne approximation du courant sauf au voisinage des intersections des parois. Le résultat est appliqué à la circulation du Pacifique considéré comme un bassin triangulaire. Les auteurs montrent que la méthode peut être appliquée dans le cas d'un bassin d'une forme quelconque, pas trop irrégulière. Comme exemple ils étudient le cas d'un bassin semi-circulaire. *M. Kiveliovitch.*

Elasticity, Plasticity

Morinaga, Kakutarô, and Nôno, Takayuki. On stress-functions in general coordinates. J. Sci. Hiroshima Univ. Ser. A. 14, 181–194 (1950).

§§ 2–5 obtain the general solution of the equilibrium equations $T^{ij}{}_{,j}+F^i=0$, $T^{ij}=T^{ji}$ for a continuous medium in Euclidean n-space. The results are obvious from the solution for $n=2, 3$ given by B. Finzi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 19, 578–584, 620–623 (1934)], work already repeated and amplified by C. Weber [Z. Angew. Math. Mech. 28, 193–197 (1948); these Rev. 10, 494], G. Peretti [Atti. Sem. Mat. Fis. Univ. Modena 3, 77–82 (1949); these Rev. 11, 557], and V. I. Bloh [Akad. Nauk SSSR. Prikl. Mat. Meh. 14, 415–422 (1950); these Rev. 12, 370]. § 6 gives a symbolic formula for the Beltrami-Michell equations expressed in terms of the stress functions. § 7 gives corresponding stress functions for $F^{ij}{}_{,j}=0$, $F^{ij}=-F^{ji}$, to be applied to electromagnetic theory when $n=4$. *C. Truesdell* (Bloomington, Ind.).

Anđelić, Tatomir. Derivation of the fundamental equations of elasticity by the Pfaffian method. Glas Srpske Akad. Nauka. Od. Prirod.-Mat. Nauka 198, 141–145 (1950). (Serbo-Croatian)

See Acad. Serbe Sci. Publ. Inst. Math. 3, 191–195 (1950); these Rev. 12, 556.

Rathgeb, Eckhardt. On the principle of virtual displacements and its application to elastic bodies. Ciencia y Técnica 116, 139–149 (1951). (Spanish. German summary)

The author has written a sound review article on the principle of virtual work and its applications to classical elasticity theory designed for the needs of those members of the engineering profession having a modest background in applied mathematics. *A. W. Sáenz.*

Föppl, O. Der Fehler der bisherigen elastizitätstheoretischen Betrachtungen. Schweiz. Arch. Angew. Wiss. Tech. 17, 171–177 (1951).

Shield, R. T. Notes on problems in hexagonal aeolotropic materials. Proc. Cambridge Philos. Soc. 47, 401–409 (1951).

The following boundary value problems are solved for elastic bodies having hexagonal aeolotropy and one or two bounding surfaces parallel to the hexagonal planes: (1) an isolated line force uniformly distributed through a plate of finite thickness and acting parallel to the faces of the plate; (2) a flat elliptic crack in a body of infinite extent with a uniform tension at infinity normal to the crack; (3) indentation of the surface of a semi-infinite body by a flat-ended, rigid punch of elliptic section; (4) an isolated force within a semi-infinite body and normal to the surface.
R. D. Mindlin (New York, N. Y.).

Lubkin, J. L. The torsion of elastic spheres in contact. J. Appl. Mech. 18, 183–187 (1951).

In the case of two elastic bodies, pressed together and then subjected to a twisting couple about an axis normal to the elliptic contact surface, the torsional compliance has been found previously on the assumption that no slip occurs on the contact surface. From a physical point of view, the result is valid only for small torques because the traction has a line singularity along the edge of the contact surface and, as a consequence, slip would be initiated there. The investigation is extended, in the present paper, by taking slip into account in the special case of spherical surfaces in contact. The problem is formulated as one of a semi-infinite body with tractions given over a portion of the surface and mixed tractions and displacements over the remainder. Solutions are found, in terms of complete elliptic integrals and other definite integrals, for the traction on the contact surface and the full range of torsional compliance.
R. D. Mindlin (New York, N. Y.).

Bouzitat, Jean. Sur l'appui lisse de deux corps solides. C. R. Acad. Sci. Paris 232, 683–685 (1951).

Explicit formulas are given for the stresses, along the axes and along the elliptic boundary of the contact surface, as deduced from the Hertz theory of contact of elastic bodies. (Courtesy of Applied Mechanics Reviews.)
R. D. Mindlin (New York, N. Y.).

Tekinalp, Bekir. On the compression of a cube between rough end-blocks. Bull. Tech. Univ. Istanbul 2, no. 2, 101–110 (1949). (English. Turkish summary)

The hypercircle method of Prager and Synge [Quart. Appl. Math. 5, 241–269 (1947); these Rev. 10, 81] is applied to the compression of a cube between perfectly rough end plates. The true modulus of elasticity is found to lie between 0.938 and 0.958 of the apparent modulus when Poisson's ratio is $\frac{1}{4}$. *D. C. Drucker* (Providence, R. I.).

Nagahara, Sigeru. Deformation of an elastic body under the influence of a gravitational field. Bull. Earthquake Res. Inst. Tokyo 20, 401–418 (1942). (Japanese. English summary)

"When a tunnel is bored, the surface of the overlying earth falls, and the mode of deformation of the earth's

surface thus caused may differ in a number of respects from that caused by external stresses, since, in the former case, the mass of the earth may play an essential part. In the present study, the author obtains expressions for displacements at any point within the earth when a cylindrical hole of circular section is bored horizontally at a certain depth. In dealing with the present problem, the plasticity of the earth and the porous effect due to contraction were disregarded. Numerical values of surface displacements were calculated for various cases, for which the dimensions of the circular tunnel and its depth are given."

Author's summary.

Wu, M. H. Lee. Analysis of plane-stress problems with axial symmetry in strain-hardening range. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2217**, 79 pp. (1950).

Simple deformation theory is employed to solve the following problems with large strain: circular membrane under lateral pressure; rotating disk with and without central hole; and an infinite plate with a circular hole stressed radially at infinity. Solutions are probably valid because the ratios of the principal stresses as well as their directions remain almost constant. *D. C. Drucker* (Providence, R. I.).

Deverall, L. I., and Thorne, C. J. Some thin-plate problems by the sine transform. J. Appl. Mech. **18**, 152–156 (1951).

This paper presents a clear and concise solution for the deflections of a rectangular plate under lateral load or edge moments using the sine transform. Nearly all edge and loading conditions are considered in a single unified treatment. The solution presented demonstrates the power of this method in giving a general solution for a class of problems. Tables of sine transforms used in the solution are presented. Examples are given of the use of these tables for obtaining solutions in two specific cases. *S. Levy.*

Coan, J. M. Large-deflection theory for plates with small initial curvature loaded in edge compression. J. Appl. Mech. **18**, 143–151 (1951).

A large-deflection solution in the form of trigonometric series is presented for several examples of buckling of a rectangular plate under axial compression. The solution is of special interest because it considers two opposite edges of the plate to be simply supported and stress free, and therefore, at liberty to warp in the plane of the plate; while the other two edges are simply supported and subjected to a uniform axial displacement. The method developed for taking account of this mixed boundary condition is quite general and should be of value in analyzing other plate problems. In addition, the author includes the effects of small deviations from initial flatness.

The general solution presented is used to derive numerical solutions for deflection and strain at the center of a square panel for several loadings of interest. These values are compared with experimental results and the good agreement confirms the theoretical derivations. Both the theory and experiment show the important effects of initial deviations from flatness and of restraint of the supported edges from transverse displacements. The author makes use of his solutions to draw interesting conclusions regarding five widely used approximate methods of estimating buckling loads from test data. He points out that the buckling load corresponds closely, if not identically, to the inflection points of both the load-deflection curve and the load-bending strain curve. *S. Levy* (Washington, D. C.).

Sen, Bibhutibhushan. Stresses due to nuclei of thermoelastic strain in a thin circular plate. Bull. Calcutta Math. Soc. **42**, 253–255 (1950).

Formulas are given for two-dimensional components of stress in a thin disk which contains an eccentric center of dilatation and has a traction-free edge. *R. D. Mindlin.*

Gerard, George. Note on bending of thick sandwich plates. J. Aeronaut. Sci. **18**, 424–426, 432 (1951).

If a characteristic length such as a buckle wave-length is of the order of the thickness of the sandwich a knowledge of the system of stresses in the core is necessary. A stress function is set up for the core, satisfying suitable conditions at the junction of the facings and core and containing four arbitrary constants that are to be determined from the further proper conditions in any given problem. This procedure has been used in treating the wrinkling problem. The author suggests that it may also have application in certain bending problems. The plate is assumed to be in a state of plane strain. Consequently it is considered to be acting as a beam or a column. *H. W. March* (Madison, Wis.).

Goran, L. A. A minimum energy solution and an electrical analogy for the stress distribution in stiffened shells. J. Aeronaut. Sci. **18**, 407–416 (1951).

A method is presented for analyzing stiffened shell structures in which "shear lag" effects are considered. The method can be applied to tapered, sweptback box beams having curved webs and cut-outs, and having straight stringers that all meet at a common point. The solution is based on the principle of minimum strain energy and uses the method of Lagrangian multipliers. The author points out that, because of the large number of simultaneous equations which result for the usual structure, an iteration procedure is probably the only practical means of obtaining a solution by analytical methods. In a presented example, convergence is found to be fairly rapid, six cycles of iteration being sufficient for good accuracy. Two electrical analogies are presented by the author. No experimental results from these electrical circuits are presented; however, the results of an iterative numerical solution of one of the equivalent electrical circuits are given and agree with those obtained directly from solution of the simultaneous equations.

S. Levy (Washington, D. C.).

Darevskiĭ, V. M. Concerning the effect on a cylindrical shell of a concentrated load. Doklady Akad. Nauk SSSR (N.S.) **75**, 7–10 (1950). (Russian)

The author gives without proof some asymptotic expansions for the displacements, the normal forces, and the bending and twisting moments in the neighbourhood of a concentrated point load on a point of a cylindrical shell.

W. H. Muller (Amsterdam).

Darevskiĭ, V. M. Concerning the effect on a cylindrical shell of certain loads. Doklady Akad. Nauk SSSR (N.S.) **75**, 169–172 (1950). (Russian)

This is an extension of the paper reviewed above. The load is now uniformly distributed along a segment of a parallel circle or a generating line. Asymptotic expansions for the displacements, the normal forces, and the bending and twisting moments in the neighbourhoods of the end points of the segments are given again without proof.

W. H. Muller (Amsterdam).

Morley, L. S. D., and Floor, W. K. G. Load distribution and relative stiffness parameters for a reinforced circular cylinder containing a rectangular cutout. Nationaal Luchtvaartlaboratorium. Amsterdam. Report S. 362, 22 pp. (1949).

The cylinder is reinforced by uniformly spaced and equal stringers of no bending stiffness and by two end rings and two rings bordering the cutout. Shear is assumed constant in each panel so that axial load in the stringers varies linearly from ring to ring. The elastic rings are supposed to have no torsional stiffness and no bending rigidity out of their planes. Six loading cases are analyzed: axial load, torsion, bending, and bending plus shear for the cutout symmetric about the neutral surface and about the extreme fiber. Total strain energy stored is minimized. Stress and stiffness calculations are made for several cases. *D. C. Drucker.*

Bax Stevens, O. Elementary derivation of the shearing stress distribution, the angle of twist and the warping in a prismatical shaft of elliptical cross section twisted by a torque. Nederl. Akad. Wetensch. Proc. Ser. B. 54, 120–129 (1 plate) (1951).

The author derives the properties listed in the title by employing (1) the constant values of τ_{zz} along $y=\text{const.}$ and of τ_{zy} along $x=\text{const.}$; (2) each stress contributes one half of the twisting moment; and (3) the deformation of all lines initially parallel to the principal axes into lines after applying the twisting couple. *D. L. Holl* (Ames, Iowa).

Nelson, Carl W. A Fourier integral solution for the plane-stress problem of a circular ring with concentrated radial loads. J. Appl. Mech. 18, 173–182 (1951).

Just as Fourier integral solutions have been obtained for infinite straight bars under various combinations of normal edge loads, similar integral solutions are obtained for a curved bar bounded by two concentric circles under radial loads. (This is really a cylindrical helix of negligible pitch with the loaded region confined to a small angular sector.) The stresses are determined from an Airy's function and the principal work of the paper is the evaluation of the infinite integrals. *D. L. Holl* (Ames, Iowa).

Kusukawa, Ken-ichi. On the theory of shock waves produced by a rigid wedge moving through an elastic medium with supersonic velocities. J. Phys. Soc. Japan 6, 163–165 (1951).

The motion produced in an elastic solid when penetrated by a rigid wedge is investigated. The theory adopted is linear (small strains). The solution contains a shock wave across which only the normal velocity component experiences a discontinuity. The results are analogous to those for the compressible fluid. *G. F. Carrier.*

Kusukawa, Ken-ichi. On the theory of shock waves produced by a rigid cone moving through an elastic medium with supersonic velocities. J. Phys. Soc. Japan 6, 166–167 (1951).

This investigation is the same as that of the above paper except for the geometry change indicated in the title.
G. F. Carrier (Providence, R. I.).

Taddei, Mario. Nota sul calcolo delle oscillazioni torsionali di un albero con massa propria. Ricerca, Napoli 1, nos. 2–3, 19–26 (1950).

Taddei, Mario. Oscillazioni torsionali smorzate di un albero con massa propria. Ricerca, Napoli 1, no. 4, 49–69 (1950).

Lotkin, O. I. On the application of Galerkin's method to the calculation of an airplane wing in flutter. Akad. Nauk SSSR. Inženernyĭ Sbornik 6, 153–160 (1950). (Russian)

Galerkin's method is applied to the differential equations of flexural-torsional flutter. The author calculates in detail approximations for the bending and torsion modes, which satisfy the primary and secondary boundary conditions, but gives only a very short indication of their use in the problem. *W. H. Muller* (Amsterdam).

Williams, John. Some developments of expansion methods for solving the flutter equations. Aeronaut. Quart. 2, 209–225 (1950).

When classical derivative theory is employed, the flutter problem can be reduced to the calculation of the common zeros in the flutter quadrant $X<0$, $Y>0$ of the bivariate polynomials $f(X, Y)$ and $g(X, Y)$ of the degree n and $(n-1)$ respectively in X and Y jointly. The expansions

$$f(X, Y)=\sum_{\substack{r,s=1\\ r\neq s}}^{n+2} A_{rs}\prod_{\substack{k=1\\ k\neq r,s}}^{n+2} L_k,\quad g(X, Y)=\sum_{\substack{r,s=1\\ r\neq s}}^{n+1} B_{rs}\prod_{\substack{k=1\\ k\neq r,s}}^{n+1} L_k,$$

where $L_k=Y-p_kX+q_k$, are calculated directly from the flutter determinant. The present author uses lines $L_k=0$ with intersections inside the flutter quadrant, close to the common zero to be determined, and shows by an example that the computational accuracy is improved compared with methods which use lines intersecting outside the flutter quadrant. In part 2 methods are given to calculate f and g in polynomial form by collocation in $\frac{1}{2}(n+1)(n+2)$ and $\frac{1}{2}n(n+1)$ distinct points respectively. When the approximations of classical derivative theory become inadequate, f and g become transcendental functions of X and can only be expanded in Y for assigned X. *W. H. Muller.*

Traill-Nash, R. W. The symmetric vibrations of aircraft. Aeronaut. Quart. 3, 1–22 (1951).

An analysis is presented in general form for the symmetric vibrations of aircraft as free bodies. It is assumed that the mass distribution of the airplane is adequately represented by lumped masses; the magnitudes and positions of the masses are selected to give as close an approximation as possible to the continuous distribution. Influence coefficients, computed with the wing-fuselage junction considered clamped, specify the elastic properties. The absolute displacements are taken as the sum of the elastic displacements and those of the wing-fuselage junction. Matrix notation is used to give a concise formulation of the eigenvalue equations. *S. Levy* (Washington, D. C.).

Press, F., Ewing, M., Crary, A. P., Katz, S., and Oliver, J. Air-coupled flexural waves in floating ice. Geophysical Research Papers No. 6, Air Force Cambridge Research Laboratories, Cambridge, Mass., 46 pp. (1950).

The authors made observations on floating ice sheets of flexural waves produced by explosions below, within and above the ice. In the first two cases the normal sequence of dispersive waves was found. In the last case they observed only a train of waves of constant frequency corresponding to a phase velocity equal to the speed of sound in air, indicating strong coupling between the air and ice waves. The authors devise a mathematical theory in which the motions

of the two fluids are governed by velocity potentials and the motion of the plate follows classical thin plate theory, with appropriate boundary conditions at the interfaces. Solutions for a sinusoidal steady state and an exponential transient point source are then studied by methods previously employed by Lamb, Sezawa, Pekeris and the authors themselves. Excellent agreement is found between theory and experiment. An appendix contains a suggestion for use of the theory to calculate the thickness of an ice sheet from measurements recorded at a distance. *R. D. Mindlin.*

Fu, C. Y. On the irrelevant roots of the Rayleigh wave equation. Acad. Sinica Science Record **2**, 388–392 (1949).

Encore une discussion du sens physique de trois racines de l'équation caractéristique de Rayleigh. À la suite de Richter [Bull. Amer. Math. Soc. **49**, 477–493 (1943); ces Rev. **5**, 140] l'auteur affirme que seule la plus grande de ces racines ait un sens physique. En supposant le milieu élastique isotrope limité par un plan et réduisant le problème à deux dimensions, il utilise les fonctions Φ et Ψ de Knott, en écrivant le déplacement infinitesimal dû à une onde sous la forme $u=\partial\Phi/\partial x+\partial\Psi/\partial z$, $w=\partial\Phi/\Phi z-\partial\Psi/\partial x$. Les fonctions Φ et Ψ vérifient les équations d'onde

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial z^2}=\frac{\rho}{\lambda+2\mu}\frac{\partial^2\Phi}{\partial t^2}=\frac{1}{V^2}\frac{\partial^2\Phi}{\partial t^2};\quad \frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial z^2}=\frac{\rho}{\mu}\frac{\partial^2\Psi}{\partial t^2}=\frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}.$$

En remplaçant Φ et Ψ par des oscillations élémentaires $\Phi=\varphi'(z)e^{i(\xi x-\omega t)}$, $\Psi=\psi'(z)e^{i(\xi x-\omega t)}$ on cherche à déterminer une onde de dilatation φ et une onde de distorsion ψ telles que leur effet combiné donne une tension nulle sur la surface, ce qui donne l'équation caractéristique de Rayleigh

$$(2\xi^2-K^2)^2-4\xi^2(\xi^2-L^2)^{1/2}(\xi^2-K^2)^{1/2}=0$$

avec $L^2=\omega^2/V^2$, $K^2=\omega^2/v^2$. D'après l'auteur les deux plus petites racines de cette équation correspondent à des angles incidents particuliers pour lesquels l'onde P réfléchie n'existe pas. Il est à noter que l'équation (10) du mémoire est erronée. *V. A. Kostitzin* (Paris).

Matuzawa, Takeo. Temperaturverlauf an der Bodenoberfläche und der Spannungszustand in der Erdkruste. I. Bull. Earthquake Res. Inst. Tokyo **20**, 20–29 (1942). (German. Japanese summary)

The author accepts as proven the very doubtful daily and annual periodicities in the occurrence of earthquakes that some authors have derived from published lists and seeks an explanation in the daily and annual variation of ground temperature. The author substitutes the formula for the two-dimensional rate of heat transfer in the equation of motion of a perfectly elastic body on the assumption of linear proportion between the tensions and the temperature changes and solves for the components of strain in a vertical plane. He concludes that the strains introduced by temperature may be comparable to those caused by atmospheric pressure changes only in the case of small islands. The original article should be consulted for the mathematics.
J. B. Macelwane (St. Louis, Mo.).

Matuzawa, Takeo. Der Temperaturverlauf an der Bodenoberfläche und der Spannungszustand in der Erdkruste. II. Verzerrung in drei Dimensionen. Bull. Earthquake Res. Inst. Tokyo 20, 265–272 (1942). (German. Japanese summary)

In a previous communication [see the preceding review] the author presented the two-dimensional theory of elastic stress distribution on the assumption of a linear relationship between the gradient of temperature change at the surface and the elastic force generated in the earth's crust. He assumed perfect elasticity and a temperature sine wave function and derived the corresponding stress-strain system for the daily and annual temperature variation. He concluded that the stresses and strains near the surface in the case of a small island would be of the same order of magnitude as those due to atmospheric pressure variations, but that at the focal depth of deep earthquakes the pressure effect would greatly predominate. In this article he generalizes the results in space coordinates, Cartesian, cylindrical, and vector. Assuming a displacement potential he obtains the stress-strain system in terms of Bessel functions.
J. B. Macelwane (St. Louis, Mo.).

Bogunović, V. Beulung der Gurtplatten von Rippenkonstruktionen. Acad. Serbe Sci. Publ. Inst. Math. **3**, 271–286 (1950).

The buckling of a decking plate with ribs, loaded along the ribs, is considered. The stress distribution before buckling is found from the stress functions in the form of series for two plane stress problems, one for the plate and one for the rib. The buckling deflection of the plate is expressed as a Fourier series, leading to a system of homogeneous equations for the coefficients. Approximations to the buckling load are obtained by equating the coefficient determinant to zero when one, two, and three coefficients are considered successively. The result is expressed in terms of an ideal system with a reduced plate width. *E. H. Lee.*

Jung, H. Ein Beitrag zur Berechnung der Knicklasten. Z. Angew. Math. Mech. **31**, 142–148 (1951). (German. English, French, and Russian summaries)

A column is considered under various end conditions. By the use of the Fourier transformation it is shown that rather complicated cases of column instability can be solved in a relatively simple manner. These cases include columns with variable cross section and elastic support of various types. An iteration process leads to satisfactorily approximate results in the less simple cases. *H. W. March.*

Duberg, John E., and Wilder, Thomas W., III. Inelastic column behavior. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2267**, i+44 pp. (1951).

The paper presents the detailed analysis of results published previously [J. Aeronaut. Sci. **17**, 323–327 (1950)]. Lateral buckling is considered of a strut of material which on loading gives the tangent modulus for the ratio of stress to strain increments, and the original elastic modulus for unloading. Deflection from the initially straight configuration can start at a range of loads from that given by the Euler load based on the tangent modulus, to that based on the elastic modulus. However, with an initial deflection, marked increase in deflection commences at the load based on the tangent modulus.

The analysis is based on a simple strut model, and a column of idealized H section with negligibly thin, but shear resistant, web. The influence of the form of the stress strain curve on the post-buckling behaviour and maximum load are also considered. *E. H. Lee* (Providence, R. I.).

Lie, Kuo-Hao. Analysis of lattice trusses. Acad. Sinica Science Record **2**, 393–401 (1949).

The author approximates a lattice truss by a continuous panel, and obtains a set of differential equations relevant to

its stress analysis. Stress distributions calculated in a specific example are shown to be in remarkable agreement with the results of a model test. *F. B. Hildebrand.*

Pöschl, Th. Über eine Anwendung der Matrizenrechnung auf die Theorie der Fachwerke. Ing.-Arch. **19**, 69–74 (1951).

This paper presents a detailed description and illustration of the construction and use of influence matrices in the analysis of statically determinate plane frameworks.
F. B. Hildebrand (Cambridge, Mass.).

Heyman, Jacques. The limit design of space frames. J. Appl. Mech. **18**, 157–162 (1951).

Printed version of an earlier report [Graduate Division of Applied Mathematics, Brown Univ., Providence, R. I., Tech. Rep. **A18-2** (1949); these Rev. **11**, 560].

Neal, B. G. The behaviour of framed structures under repeated loading. Quart. J. Mech. Appl. Math. **4**, 78–84 (1951).

Framed structures are considered subjected to several loads varying independently between prescribed maximum and minimum values. Shake down is said to occur if a state of residual stress is reached which enables all further variations of the loads to be supported without additional plastic flow. The theorem is proved that if any system of residual moments can be found permitting this, then the structure will shake down, though not necessarily with this particular residual moment distribution. An ideal moment-curvature relation is assumed for each member involving linear elasticity until the limit moment is reached at which unrestricted bending can occur. *E. H. Lee* (Providence, R. I.).

Stüssi, F. Die Grundlagen der mathematischen Plastizitätstheorie und der Versuch. Revista Acad. Ci. Madrid **44**, 123–138 (1950).

The main portion of the paper is concerned with the results of tests on thin-walled aluminum alloy tubes. By the application of an interior pressure and an axial compression a state of pure hoop tension in the plastic range is produced. On this, further stresses are superimposed by changing the interior pressure and axial compression and applying a torque. The axial and circumferential extensions and the angle of twist caused by these additional stresses are measured. Under pure hoop tension, the plastic compressive strain in the axial direction is found to be less than half the plastic tensile strain in the circumferential direction. The author concludes from this that the plastic deformation is not volume-preserving. Actually, a more likely explanation would seem to be that the material was not isotropic. This seems to be confirmed by some of the subsequent tests. The author claims that his experiments prove the basic assumptions of the mathematical theory of plasticity to be in error. To the reviewer this claim appears to be far too sweeping because the only theories with which the author compares his results are of the "deformation" or "finite" type. It is a well established fact that theories of this type agree with experiments only if the principal axes of stress remain fixed with respect to the material and if the ratios between the principal components of the stress deviation are preserved during the test. Since the author's tests are not conducted in this manner, the discrepancy between the experimental results and the predictions furnished by the finite theories used by the author is not surprising. It would be interesting to confront the author's experimental results with a "flow" or "incremental" theory which takes account of the initial anisotropy of the material. *W. Prager.*

Mii, Hisao. Some notes on the plastic deformation of hollow spheres with large strain. J. Jap. Soc. Appl. Mech. **3**, 133–139 (1950).

Wu, M. H. Lee. Linearized solution and general plastic behavior of thin plate with circular hole in strain-hardening range. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **2301**, 41 pp. (1951).

The annular plate is pulled with uniform tension at the periphery to give strains sufficiently large to neglect elastic strains. Assuming a deformation type stress-strain law, differential equations are obtained which can be linearized with good accuracy to give a simple solution for a power law approximation to the stress-strain curve. Good agreement is obtained with previous solutions of the nonlinearized equations. A parameter which determines the variation of stress ratios at each point is indicated. This variation is shown to be small, so that the solution will be a good approximation to that based on an incremental type law. The ideally plastic approximation gives good results for strain distribution, but not for stress. Curves of stress and strain distribution for various parameters are given.
E. H. Lee (Providence, R. I.).

Malvern, L. E. The propagation of longitudinal waves of plastic deformation in a bar of material exhibiting a strain-rate effect. J. Appl. Mech. **18**, 203–208 (1951).

Reviewed earlier as a report [Graduate Division of Applied Mathematics, Brown Univ., Providence, R. I., Tech. Rep. **A11-39** (1949); these Rev. **12**, 374].

Kuranishi, Masatsugu. The buckling stress of thin cylindrical shell under axial compressive load, forming axial-symmetrical deformation. J. Jap. Soc. Appl. Mech. **3**, 139–144 (1950).

Plastic buckling loads are found by assuming: (a) The cylinder buckles in axial symmetric outward and lowered waves, each of which is symmetric about its crest; (b) the tangent modulus applies for outward and the elastic modulus for inward buckling in the standard beam-on-elastic-foundation type of ordinary differential equation. Good agreement with experiment is found when buckling occurs well in the plastic range. *D. C. Drucker.*

Eshelby, J. D., Frank, F. C., and Nabarro, F. R. N. The equilibrium of linear arrays of dislocations. Philos. Mag. (7) **42**, 351–364 (1951).

The authors consider the problem of determining the equilibrium positions of a series of parallel and like dislocations of infinite length lying in the plane $y=0$ and acted upon by a given applied stress. Since the dislocations repel each other inversely as the distance between them, the problem is similar to that of a series of line charges acted upon by an electric field. Hence, the problem becomes one of determining the solutions of

$$(1) \qquad \sum_{i=1,\, i\neq j}^{n} \frac{A}{x_j - x_i} + P(x_j) = 0, \qquad j=1, 2, \cdots n,$$

where A is an appropriate constant and P is the proper component of the applied stress. The relation of this problem to orthogonal polynomials has been studied by Stieltjes [Acta Math. **6**, 319–324 (1885), p. 321] and Szegö [Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ.,

v. 23, New York, 1939; these Rev. 1, 14]. The present authors introduce the polynomial $f(x)=\prod_{i=1}^{n}(x-x_i)$, and replace the equilibrium relations (1) by

$$(2)\qquad f(x_j)=0,\quad \tfrac{1}{2}f''(x_j)/f'(x_j)+P(x_j)=0,\quad j=1,\cdots,n.$$

In order to determine the permissible x_j, the authors have used a suggestion of H. Heilbronn that the following differential equation be studied:

$$(3)\qquad f''(x)+2P(x)f'(x)+q(n,x)f(x)=0.$$

By choosing $q(n,x)$ so that the differential equation has a polynomial solution of the nth degree with real distinct roots and such that $q(n,x)$ does not have a pole at any of these roots, the conditions (2) are satisfied. The modifications of (2), (3) for the existence of "locked" dislocations is considered and the case $P(x)=\text{constant}$ is studied in some detail. This last case is of physical interest. *N. Coburn.*

Bishop, J. F. W., and Hill, R. A theory of the plastic distortion of a polycrystalline aggregate under combined stresses. Philos. Mag. (7) **42**, 414–427 (1951).

Ein allgemeingültiger Zusammenhang zwischen Spannung und plastischer Deformation wird für polykristalline Aggregate unter der Annahme hergeleitet, dass es sich blos um Gleitungen entlang bevorzugter Gleitebenen handelt. Für die Fliessgrenze von einem sich in einem bestimmten Verfestigungszustand befindenden Metalle hat man $f(\sigma_{ij}')=c$, wo c eine Konstante ist und die σ_{ij}' erhält man aus den Spannungskomponenten σ_{ij}, wenn man von denen den hydrostatischen Druck subtrahiert. Weiter nimmt man an, dass unter Vernachlässigung der elastischen Deformation die Gleichung $d\epsilon_{ij}=h(\partial g/\partial\sigma_{ij})df$ bestehen soll, wo die ϵ_{ij} die Deformationskomponenten, g das plastische Potential und h einen Proportionalitätsfaktor bedeuten, die letzteren zwei Grössen sind nur Funktionen der σ_{ij}'. Meistens nimmt man weiter an, dass $g=f$ ist, daraus kann man einige Extremalprinzipe herleiten.

In der vorliegenden Arbeit leiten die Verfasser diese Prinzipien, d. h. das Prinzip der maximalen Arbeit und das der minimalen Gleitung unter der erwähnten Annahme bezüglich der auftretenden Deformationen erstens für Einkristalle und daraus für mikrokristalline Aggregate her, womit die Benützung der obigen Gleichung gerechtfertigt wird. Da die genaue Berechnung der durch die Funktion f definierten Hyperfläche schwierig ist, so werden noch zum Schluss zwei Flächen berechnet, zwischen denen die liegen muss und ausserdem wird noch die Abhängigkeit dieser Fläche von der Deformations-Vorgeschichte des fraglichen Materials besprochen. *T. Neugebauer* (Budapest).

Leibfried, G. Zur atomistischen Theorie der Elastizität. Z. Physik **129**, 307–316 (1951).

Bekannterweise hat zuerst Born die elastischen Konstanten eines festen Körpers auf atomare Daten auf dem Wege zurückgeführt, dass er in dem atomtheoretischen Ausdruck für die Spannungen solche Verschiebungen der einzelnen Atome einsetzt, die vom Standpunkte der Elastizitätstheorie aus einer homogenen Verzerrung entsprechen. Daraus folgt ein Zusammenhang zwischen Spannungen und Verzerrungen d. h. also die elastischen Konstanten. Eine zweite Methode von Born beruht auf der Theorie der Ausbreitung von ebenen Wellen im Kristall. Besonders gegen die erste Methode sind in den vergangenen Jahren Bedenken geäussert worden; so bezweifelt P. S. Epstein [Physical Rev. (2) **70**, 915–922 (1946); diese Rev. **9**, 119] die Richtigkeit der ganzen Methode, besonders hinsichtlich der Cauchyschen Relationen. Andere Verfasser behaupteten, dass das Bornsche Resultat nur zufällig richtig herauskommt usw.

Diesen Behauptungen gegenüber beweist der Verfasser, dass die Berechnungen von Born richtig sind. Zu diesem Zwecke benützt er ein Verfahren, das schon von Born angedeutet, jedoch nicht ausgeführt wurde. Analog zu den früheren Berechnungen wird die Existenz einer potentiellen Energie zwischen je zwei Teilchen im Gitter angenommen. Diese Energie hängt nur von dem Abstand ab; es werden also nur Zentralkräfte berücksichtigt. Die auftretenden Verschiebungen sollen ausserdem klein sein, so dass man nach ihnen entwickeln kann. Zur elastischen Theorie wird dann auf dem Wege übergegangen, dass man nur sehr langsam mit dem Orte veränderliche Verschiebungen behandelt und die durch ein Verschiebungsfeld beschreibt. Die Bewegungsgleichungen der Atome folgen dann aus den Euler-Lagrangeschen Differentialgleichungen und durch Vergleich derselben mit den Gleichungen der phänomenologischen Elastizitätstheorie folgen endlich die elastischen Konstanten. Zuerst wird nur das aus einer Atomsorte aufgebaute primitive Translationsgitter und dann der allgemeine Fall besprochen. Im ersteren bestehen die Cauchyschen Relationen immer, im letzteren unter gewissen schon von Born besprochenen Voraussetzungen.

T. Neugebauer (Budapest).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Glaser, W. The refractive index of electron optics and its connection with the Routhian function. Proc. Phys. Soc. Sect. B. **64**, 114–118 (1951).

In classical dynamics with a Lagrangian $L=T(q,\dot q)-V(q)$, $T=\frac{1}{2}\sum a_{jk}\dot q_j\dot q_k$, there are three equivalent stationary principles commonly written (A) $\delta\int Ldt=0$, (B) $\delta\int Tdt=0$, (C) $\delta\int(E-V)^{\frac{1}{2}}ds=0$, the last two being very closely allied to one another. In electron optics $L=L(q,\dot q)$, but not of the above special form, and (B) and (C) are generalised to (B′) $\delta\int\{\sum\dot q_k\partial L/\partial\dot q_k\}dt=0$, (C′) $\delta\int\mu(q,q',E)ds=0$, where t in (B′) is such that $\sum\dot q_k\partial L/\partial\dot q_k-L=E$ (the constant of energy) and (C′) is the result of getting rid of t from (B′) and using a general parameter s ($q'=dq/ds$); μ is the refractive index. In a previous paper [Z. Physik **80**, 451–464 (1933)] the author obtained (C′) with $\mu=(L+E)/v$ where $v=ds/dt$, and in the present paper defends his argument against criticism by W. Ehrenberg and R. E. Siday [Proc. Phys. Soc. Sect. B. **62**, 8–21 (1949)]. In the opinion of the reviewer the result follows immediately from the well known (B′) and its deduction is not facilitated by the reasoning of the present paper, there being on p. 116 several misprints and formulae difficult to understand, while the derivation of equation (20) (in which $u=s$ and so has different terminal values for the curves of the family considered) from (9) (in which the terminal values of u are fixed) is not satisfactory, although the final result is certainly correct. Elimination of the velocities corresponding to cyclic coordinates from L by means of the Routhian procedure is discussed and applied to the motion of an electron in an axially symmetric electromagnetic field; the determination of the paths is reduced to

finding the rays in an isotropic two-dimensional optical medium. *J. L. Synge* (Dublin).

Clemmow, P. C. A method for the exact solution of a class of two-dimensional diffraction problems. Proc. Roy. Soc. London. Ser. A. **205**, 286–308 (1951).

Abstract based on the author's summary: Several problems of electromagnetic diffraction theory have recently been solved by application of the Wiener-Hopf technique to the underlying integral equations for the screen currents, as the diffraction of a plane wave by a half-plane, two parallel half-planes, and an infinite set of parallel half-planes. An alternative and [in the author's opinion] simpler approach to all these problems is given in the paper under review. The author bases his theory on a representation of the scattered field as an angular spectrum of plane waves. This representation leads to a pair of dual integral equations instead of a single integral equation of the earlier method. Several points of general interest in diffraction theory are discussed, including the question of the nature of the singularities at a sharp edge [the author claims that Copson has "clarified" a question that "is a matter of some debate"], and it is shown that the solution for an arbitrary (three-dimensional) incident field can be derived from the corresponding solution for a two-dimensional incident plane wave. *C. J. Bouwkamp* (Eindhoven).

Aoki, Tosio. On the diffraction of electromagnetic waves by screens and holes of perfect conductors. I. On a dual relation between the diffractions of electromagnetic waves by screens and that by holes of perfect conductors. J. Phys. Soc. Japan **4**, 183–185 (1949).

This is an attempt to prove the rigorous Babinet's principle of electromagnetic diffraction theory, based on Sommerfeld's concept of double space. The author is apparently unaware of the precise results of Copson [Proc. Roy. Soc. London. Ser. A. **186**, 100–118 (1946); these Rev. **8**, 179] and Meixner [Z. Naturforschung **3a**, 506–518 (1948); these Rev. **11**, 141]. *C. J. Bouwkamp.*

Aoki, Tosio. On the diffraction of electromagnetic waves by screens and holes of perfect conductors. II. Approximate formulas and their applications. J. Phys. Soc. Japan **4**, 186–191 (1949).

This paper presents an assembly of formulas relating to the Kirchhoff-Kottler formulation of Huygens' principle in electromagnetic diffraction theory. The diffracted field at great distances behind the hole is evaluated for circular, elliptical, and rectangular holes. *C. J. Bouwkamp.*

Horton, C. W., and Karal, F. C., Jr. On the diffraction of a plane electromagnetic wave by a paraboloid of revolution. J. Appl. Phys. **22**, 575–581 (1951).

The problem of the diffraction of a plane electromagnetic wave by a paraboloid of revolution is attacked using paraboloidal wave functions. The calculations are involved and it was not considered feasible to complete them in the general case, although a further treatment of the case of a perfectly conducting paraboloid is promised when numerical values of the paraboloidal wave functions are available. For the case of a plane wave normal to the axis the authors are able to calculate the ratio of reflected to incident amplitude for points on the axis. *E. Pinney.*

Löwdin, Per-Olov. A note on the method of steepest descents with a remark on T. Ljunggren's paper "Contributions to the theory of diffraction of electromagnetic waves by spherical particles." Ark. Fys. **2**, 367–370 (1950).

Remark on a paper by Ljunggren [Ark. Mat. Astr. Fys. **36A**, no. 14 (1949); these Rev. **11**, 294; cf. also Ark. Fys. **1**, 1–18 (1949); these Rev. **11**, 705] to the effect that simpler asymptotic expressions may be obtained by developing the integrand around points not coinciding with the saddle points but in the immediate vicinity of them. The author illustrates this by an example in which a saddle point is near one of the limits of integration [in Eq. (2), F should be replaced by G]. *C. J. Bouwkamp* (Eindhoven).

Karp, Samuel, and Sollfrey, William. Diffraction by a dielectric wedge with application to propagation through a cold front. New York University, Washington Square College, Research Group, Research Rep. No. **EM-23**, ii+45 (1950).

The authors are concerned with the diffraction of plane-polarized electromagnetic waves by an infinite dielectric wedge whose dielectric constant differs but slightly from that of free space. The incident wave propagates in a direction perpendicular to the edge with the electric vector parallel to the edge, so that the diffraction problem is essentially a scalar two-dimensional one. In addition, the wedge is placed on a perfectly conducting infinite plane so as to simulate a situation that is of practical importance in radio wave propagation over plane earth through a cold front. The reflected and refracted waves are first derived by geometrical optics under the condition $\frac{1}{2}\pi<2\alpha\pm\varphi<\pi$, so as to limit the number of refracted waves to two (α is the slanting angle of the front, φ the angle of incidence with the horizontal). There is a similar discussion for incident waves propagating inside the wedge towards the front. Secondly, the fields are expanded in powers of the difference in the dielectric constant of the wedge and free space, only first-order terms being retained. These terms represent a diffraction effect proper, while the zero-order terms in the expansion correspond to the geometrical-optics field. The angular pattern of the diffracted cylindrical waves is found by stationary-phase methods. The results are applied to airplane reception of electromagnetic signals when the airplane crosses the cold front. *C. J. Bouwkamp* (Eindhoven).

Rhodes, D. R. Theory of axially slitted circular and elliptic cylinder antennas. J. Appl. Phys. **21**, 1181–1188 (1950).

The author studies the diffraction of plane-polarized electromagnetic waves by axially slotted circular and elliptic cylinders. The coefficients in the Bessel-function expansions of the Hertzian potentials are calculated according to a method suggested by Sommerfeld [Partial Differential Equations . . . , Academic Press, New York, 1949, pp. 29–31; these Rev. **10**, 608]. Simple results are obtained for slots of vanishing width (slits). Circular cylinders containing one or two diametrically opposed slits are treated in some detail. [In the first line following Eq. (20c), Newman should read Neumann.] *C. J. Bouwkamp* (Eindhoven).

Twersky, Victor. On the scattered reflection of electromagnetic waves. New York University, Washington Square College, Research Group, Research Rep. No. **EM-26**, ii+82 pp. (1951).

The author summarizes his paper as follows. "The non-specular reflection of plane electromagnetic waves of arbi-

trary polarization by certain perfectly conducting surfaces composed of either semicylindrical or hemispherical bosses on an infinite plane is analyzed. Solutions in terms of eigenfunctions for the problem of the single boss on an infinite plane and a plane wave at an arbitrary angle of incidence are derived and extended, subject to the single scattering hypothesis, to obtain the far field solutions for certain small finite patterned distributions and both small finite and infinite uniform random distributions of bosses small compared with the wavelength. The results for the various cases are then compared in the plane of incidence and similarities between the analogous expressions for the distributions of semicylinders and hemispheres are noted. Expressions are obtained for the ratios of the reflected intensities and radial energy flux polarized parallel and perpendicular to the plane of incidence, as well as for the total intensity and radial energy flux for the case where the incident wave is unpolarized. It is found that for certain values of the parameters the reflected radiation may consist only of either the specular or the scattered contributions, while for other values of the parameters one of the scattered contributions, either the parallel or perpendicular component, may vanish. The results also indicate the occurrence of an extremum in the reflected radiation in the vicinity of the specular angle of reflection which for certain ranges of the parameters for the small finite distributions may be a minimum rather than a maximum. For these cases there is also some critical angle of incidence (not necessarily $\pi/2$ or grazing incidence) for which the reflection at the specular angle is completely specular. The analogous distributions of cylinders and spheres are also considered." *M. J. O. Strutt.*

Shmoys, Jerry. On the definition of virtual height. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. **EM-27**, ii+43 pp. (1951).

The author presents a critical discussion of the concept of virtual height relating to the structure of the ionosphere. He compares the geometrical-optics and wave-theoretical definitions of virtual height for various simple types of charge distribution, including the Epstein and parabolic layers. The author's fully worked out examples indicate that the relation between virtual height and frequency derivative of phase is not valid when the reflected wave contains more than one pulse, and that this derivative cannot then be interpreted as the time delay of any one of the pulses. The problem of evaluating the time delay of each reflected pulse in the latter case remains open.

C. J. Bouwkamp (Eindhoven).

Kahan, T., et Eckart, G. Théorie de la propagation des ondes électromagnétiques dans le guide d'onde atmosphérique. Ann. Physique (12) 5, 641–705 (1950).

The first fifteen pages of this paper have been published under a different title [J. Phys. Radium (8) **10**, 333–341 (1949); these Rev. **11**, 565]. The remainder is apparently the promised paper mentioned in the review of the cited paper. There is no reference to the earlier paper in the bibliography of the paper under review.

C. J. Bouwkamp (Eindhoven).

Eckart, Gottfried. Über die Strahlung eines magnetischen Dipols in kugelförmig geschichteter Atmosphäre. Arch. Elektr. Übertragung **5**, 113–118 (1951).

A magnetic dipole is located in a point on the z-axis. It has a distance r from the origin and its direction is parallel to the z-axis. The dielectric constant ϵ of the surrounding medium is assumed to be a function of the distance r from the origin only, according to the law $\epsilon(r)=\alpha+\beta/r$ where α and β are constants. The author gives two different methods of computing the radiation field. One is based on the theory of Green's function, the other one gives a direct connection with the wave function $R^{-1}e^{-ikR}$. The equating of both expansions gives a relation between Bessel and confluent hypergeometric functions. *F. Oberhettinger.*

Goubau, Georg. Surface waves and their application to transmission lines. J. Appl. Phys. **21**, 1119–1128 (1950).

Abstract based on the author's summary: The applicability of nonradiating surface waves for transmission lines is investigated. One type of wave, first studied by Sommerfeld [Ann. Physik **67**, 233–290 (1899)], is guided by a conductor of finite conductivity. Its practical application is restricted because the extension of the field is too large for moderate frequencies. A second type of wave is guided by a conductor coated with a dielectric layer or of surface otherwise modified. In this case the extension of the field can be controlled by the surface modification. A thorough numerical discussion is given. *C. J. Bouwkamp.*

Tai, C. T. The effect of a grounded slab on the radiation from a line source. J. Appl. Phys. **22**, 405–414 (1951).

The author calculates the effect of a grounded dielectric slab on the radiation from a line source. The principal part of the electric field above the slab is derived in two ways by the techniques of Fourier transformation and saddle-point approximation. This radiation field is discussed in terms of geometrical optics. Conditions of the existence of surface waves and propagating modes in the slab are studied. [The radiation condition is often referred to but nowhere explicitly formulated. As usual, the difficult question as to the accuracy given by the principal term of the saddle-point evaluation is left unanswered.] *C. J. Bouwkamp.*

Spencer, Domina Eberle. Separation of variables in electromagnetic theory. J. Appl. Phys. **22**, 386–389 (1951).

Bei der Separation der Variablen bei elektromagnetischen Aufgaben tritt die Schwierigkeit auf, dass an Stelle einer skalaren unabhängigen veränderlichen Grösse jetzt ein Vektor auftritt. Hierdurch entstehen bei der Transformation auf krummlinige rechtwinklige Koordinaten andere Ausdrücke. Verf. will nun untersuchen, in welchen Koordinatensystemen die Separation im allgemeinen Falle durchgeführt werden kann. Sie geht von den Maxwellschen Differentialgleichungen aus und gelangt aus ihnen auf eine Wellengleichung für einen Vektor als unabhängige Variable. Diese Wellengleichung wird auf allgemeine krummlinige Koordinaten transformiert. Verf. gibt zwei notwendige Bedingungen für die Möglichkeit der Separation der unabhängigen Variablen an. Es zeigt sich, dass die zweite notwendige Bedingung nur für rechtwinklige Koordinaten, Kreiszylinderkoordinaten und Kugelkoordinaten erfüllt ist. Die erste Bedingung ist jedoch nur vollständig für rechtwinklige Koordinaten erfüllt. Eine einzige der drei simultanen Wellengleichungen ist im Falle der Kreiszylinderkoordinaten separierbar. Verf. führt diese Separation durch und betrachtet in einer Tabelle folgende Fälle für die allgemeine Wellengleichung: rechtwinklige Koordinaten, kreiszylindrische Koordinaten, elliptische Zylinderkoordinaten, parabolische Zylinderkoordinaten, Kugelkoordinaten, abgeplattete und verlängerte Sphäroidkoordinaten und parabolische Koordinaten. Als Sonderfälle betrachtet sie

zylindrische Felder, welche von einer Koordinaten unabhängig sind und sodann axial symmetrische Felder.

M. J. O. Strutt (Zürich).

Kline, Morris. An asymptotic solution of Maxwell's equations. New York University, Washington Square College, Mathematics Research Group, Research Rep. No. **EM-24**, ii+43 pp. (1950).

Duhamel's principle relates the electromagnetic field due to an arbitrary electric-charge distribution with harmonic time behavior ("harmonic solution" of Maxwell's equations) to the field created by the same charge suddenly placed in space at time zero ("pulse solution") [cf. R. K. Luneberg, same report series, No. **EM-14** (1949); these Rev. **11**, 630]. It follows from this principle that for the harmonic solution any component of the field vectors approaches the form $u \exp(-i\omega t)$ when transient phenomena have died away. An asymptotic expansion of u can be given in powers of the wavelength. The coefficients are simply related to the discontinuities in the time derivatives of the pulse solution. The author indicates a possible way to derive these discontinuities without full knowledge of the pulse solution, which is of importance because both the pulse solution and the harmonic solution are not readily evaluated in practical cases. A system of recursive ordinary differential equations for the discontinuities is used. The paper is self-contained though there is some overlapping with Luneberg's report cited above.

C. J. Bouwkamp (Eindhoven).

Price, A. T. Electromagnetic induction in a semi-infinite conductor with a plane boundary. Quart. J. Mech. Appl. Math. **3**, 385–410 (1950).

Verf. erwähnt zunächst die früheren Untersuchungen über die gleiche Aufgabe. Er geht von den allgemeinen Maxwellschen Gleichungen aus und sucht Lösungen dieser Gleichungen, welche Produkte einer Funktion von z und t einerseits und von x und y andererseits sind. Hierbei ist die Grenzebene die (x, y)-Ebene. Für diese beiden Funktionen ergeben sich einfache Differentialgleichungen, deren Elementarlösungen vom Verf. angegeben und diskutiert werden. Sie geben Anlass zu gewissen Zeitkonstanten und Feldformen im Leiter. Sodann geht Verf. auf ein periodisches induzierendes Feld über. Er berechnet die induzierten periodischen Ströme im Leiter und stellt dieselben graphisch dar. Zum Schluss betrachtet Verf. diesen Leiter als Grenzfall eines grossen Kugelleiters. Weiterhin betrachtet er aperiodische induzierende Felder und wendet in diesen Fällen die operatorische Lösungsmethode an.

M. J. O. Strutt (Zürich).

Millsaps, Knox, and McPherson, J. C. The oscillations of magnetic suspensions. J. Appl. Phys. **22**, 429–432 (1951).

The authors treat the one-dimensional problem in which a magnetic pole oscillates between two others in a medium providing resistance proportional to the square of the velocity. For zero resistance the motion may be expressed in terms of elliptic integrals. When resistance is present but small, an approximate numerical treatment is given. Numerical results given for zero damping and small damping are sufficiently similar to suggest the usefulness of standard quasi-linear methods in such problems.

E. Pinney.

Gross, E. P. Plasma oscillations in a static magnetic field. Physical Rev. (2) **82**, 232–242 (1951).

Die Schwingungen von einem sich in einem statischen magnetischen Felde befindenden Plasmas werden berechnet, wobei die auftretenden Amplituden als klein vorausgesetzt werden. Die angenäherte Behandlung dieses Problems ist aus der Theorie der Ausbreitung von elektromagnetischen Wellen in der Ionosphäre bekannt; zur strengen Lösung der Frage geht der Verfasser aus der Boltzmannschen Differentialgleichung

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f+\frac{e}{m}\left(\mathbf{E}+\frac{\mathbf{v}\times\mathbf{H}_0}{c}\right)\cdot\frac{\partial f}{\partial \mathbf{v}}=\left(\frac{\delta f}{\delta t}\right)_{\mathrm{coll}}$$

für die Verteilungsfunktion $f(\mathbf{x}, \mathbf{v}, t)$ aus. Die ersten zwei Glieder auf der linken Seite entsprechen der eigentlichen Kontinuitätsgleichung, die hier mit den infolge der Lorentzkraft auftretenden Änderungen von f und auf der rechten Seite ausserdem mit den von den Zusammenstössen verursachten Änderungen ergänzt ist. Weiter folgt aus den Maxwellschen Differentialgleichungen

$$\nabla\times(\nabla\times\mathbf{E})=-(\partial^2\mathbf{E}/c^2\partial t^2)-(4\pi\partial\mathbf{j}/c^2\partial t),$$

wo $\mathbf{j}=e\int\mathbf{v}fd\mathbf{v}$ ist. Diese zwei Systeme sind die gekoppelten nichtlinearen Integro-Differentialgleichungen des Problems. Zur Lösung setzt der Verfasser erstens $f=f_0(\mathbf{v})+f_1(\mathbf{x}, \mathbf{v}, t)$, wo $f_1 \ll f_0$ ist und führt dann ein Polarkoordinatensystem (ρ, δ) im Geschwindigkeitsraum ein, mit dessen Hilfe die vereinfachte Boltzmannsche Gleichung tatsächlich gelöst werden kann. Zur weiteren Besprechung dieser Lösung und besonders zur Berechnung der periodischen Lösungen wird die bekannte Entwicklung nach Besselschen Funktionen $e^{iz\sin\delta}=\sum_{n=-\infty}^{+\infty}J_n(z)e^{in\delta}$ benutzt. Setzt man die erhaltene Lösung in das etwas vereinfacht geschriebene System der ursprünglichen Integro-Differentialgleichungen ein, so folgt endlich die Dispersionsgleichung, also der Zusammenhang zwischen Frequenz und Wellenlänge als eigentliche Lösung des Problems. Die dabei auftretenden Koeffizienten werden mit Hilfe von Besselschen Funktionen ausgedrückt. Die Rechnungen werden für den Fall einer durch eine Diracsche δ-Funktion ausgedrückte, also ein sehr scharfes Maximum besitzende und eine Maxwellsche Geschwindigkeitsverteilung durchgeführt.

Als wichtigstes Ergebnis folgt, dass das Spektrum in der Umgebung der Vielfachen der Larmorfrequenz Lücken aufweist und das ausserdem in der Umgebung der Debyeschen Wellenlänge eine starke Dämpfung auftritt. Keines dieser Resultate folgt aus der einfachen die Maxwellschen Transportgleichungen benutzenden Theorie, die jedoch sonst qualitativ richtige Ergebnisse liefert.

T. Neugebauer.

Bohm, David, and Pines, David. A collective description of electron interactions. I. Magnetic interactions. Physical Rev. (2) **82**, 625–634 (1951).

Zur Behandlung der in einem Elektronengas oder Plasma auftretenden Wechselwirkungen wird eine neue Methode entwickelt, die nicht von dem Modell der freien Partikeln ausgeht (in der man die Wirkung auf ein herausgegriffenes Elektron auf die Weise berücksichtigt, dass man den Einfluss der übrigen verschmiert [z. B., nach Thomas-Fermi oder Hartree-Fock] und das die Korrelationen zwischen den Orten der Elektronen ganz vernachlässigt), sondern das organisierte Verhalten eines Elektronengases von grosser Dichte betrachtet, demzufolge die sogenannten Plasmaschwingungen im System auftreten. In dieser kollektiven Betrachtungsweise eines Elektronengases berücksichtigt man zuerst eine Fourierkomponente des Feldes und nimmt an, dass für kleine Amplituden das Feld als Summe solcher Glieder darstellbar ist. Die Bedingung der sich aufrechterhaltenden Schwingungen ist, dass das von den Verrückungen

der Partikel verursachte Feld mit dem erwähnten übereinstimmt. Aus dieser Bedingung folgt die Dispersionsgleichung.

Ausgegangen wird aus der sich auf das Einzelpartikelmodell beziehenden Hamiltonschen Funktion, die aus drei Teilen besteht, die sich auf die Partikel, auf das Strahlungsfeld und die Wechselwirkung von denen beziehen. Das Problem ist dann solch eine kanonische Transformation zu finden, dass die neuen Veränderlichen des Feldes unabhängig von den neuen kanonischen Koordinaten der Partikel werden und mit der charakteristischen Frequenz des organisierten Verhaltens schwingen. Diese Berechnungen werden vom Verfasser unter einigen berechtigten Vernachlässigungen auch nach der klassischen Theorie und auch nach der Quantenmechanik durchgeführt, wobei man in beiden Fällen eine Dispersionsformel erhält. Die quantenmechanische Formel

$$\omega^2=\omega_p^2+c^2k^2+(4\pi e^2/mL^3) \times\{\sum_{ik\mu}(k^2/\mathbf{P}_i\cdot\boldsymbol{\varepsilon}_{k\mu})^2/m^2\}\{1/(\omega-\mathbf{k}\cdot\mathbf{P}_i/m)^2-\hbar^2k^4/4m^2\},$$

in der ω_p die Kreisfrequenz der gewöhnlichen Plasmaschwingungen bedeutet, $k=2\pi/\lambda$ ist, $\boldsymbol{\varepsilon}_{k\mu}$ mit der Polarisationsrichtung der transversalen Schwingungen zusammenhängt und alle anderen Buchstaben die gewohnte Bedeutung haben, ist nur unwesentlich von der klassischen verschieden. Bemerkt sei noch, dass sich die Rechnungen auf transversale Schwingungen beziehen und ausserdem nur magnetische Wechselwirkungen berücksichtigt werden; für die elektrischen soll das in einer weiteren Arbeit geschehen.

T. Neugebauer (Budapest).

Lewis, I. A. D. A symbolic method for the solution of some switching and relay-circuit problems. Proc. Inst. Elec. Engrs. Part I. 98, 181–191 (1951).

The author uses a new notation to solve switching problems of the sort that Shannon solved using Boolean algebra [Trans. Amer. Inst. Elec. Engrs. 57, 713–723 (1938)].

E. N. Gilbert (Murray Hill, N. J.).

Quantum Mechanics

Brdička, Miroslav. Remark on the proper Lorentz transformation of the Dirac equations. Rozpravy II. Třídy České Akad. 60, no. 12, 7 pp. (1950). (Czech)

The aim of the present investigation is the construction of a more general proof of the Lorentz invariance of Dirac's equations for a single electron in the absence of any external electromagnetic field. As is well known, Dirac proved the covariance of his equations with respect to a Lorentz transformation of coordinates $x_i'=\sum_{k=1}^4 a_{ik}x_k$, where x_1, x_2, x_3 are Cartesian spatial coordinates and $x_4=it$, by showing that the equations are covariant with respect to simple rotations in space and with respect to the special Lorentz transformation. The author deduces now the transformations of Dirac's wave functions without any restrictions on the coefficients a_{ik} other than the orthogonality conditions $\sum_i a_{ik}a_i=\delta_k$, where δ_k stands for the Kronecker δ, and the normalization condition requiring that the determinant $\|a_{ik}\|=+1$. Several properties of such transformations are investigated in some detail.

Z. Kopal (Manchester).

Klein, Martin J., and Smith, Robert S. A note on the classical spin-wave theory of Heller and Kramers. Physical Rev. (2) 80, 1111 (1950).

A discussion and clarification of some points in the spin-wave theory of ferromagnetism proposed by G. Heller and H. A. Kramers [Nederl. Akad. Wetensch., Proc. 37, 378–385 (1934)].

N. Rosen (Chapel Hill, N. C.).

Laurikainen, K. V. Über die Gravitationsenergie des materiefreien elektromagnetischen Feldes. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 68, 45 pp. (1950).

L. Rosenfeld [Z. Physik 65, 589–599 (1930)] has shown that for a quantized electromagnetic field the gravitational energy, calculated by means of the linear gravitational field equations, diverges. The author applies to this problem the procedure of T. Gustafson [Kungl. Fysiografiska Sällskapets i Lund Förhandlingar 15, no. 28, 277–288 (1945); these Rev. 7, 180] based on the method of M. Riesz [Acta Math. 81, 1–223 (1949); these Rev. 10, 713] and again obtains a divergent result. However, by using a procedure suggested by W. Pauli, he is able to remove this difficulty. *N. Rosen.*

Nilsson, S. B., and Laurikainen, K. V. On the gravitational self-energy of light. Physical Rev. (2) 80, 291–292 (1950).

Using covariant perturbation theory, the self-energy of a photon is calculated, supposing the photon to interact with a quantized and linearized gravitational field. The result is a highly singular expression which may be made to vanish by a suitably chosen regularization procedure. A more detailed account will be published later. *F. J. Dyson.*

Rumer, Yu. B. Action as coordinate of a space. IV. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 21, 454–461 (1951). (Russian)

This is the fourth of a series of papers [same Žurnal 19, 86–94, 207–214, 868–875 (1949); these Rev. 10, 580; 11, 401] expounding the author's five-dimensional unified field theory. The main part of the paper is concerned with the formal structure of the theory in the weak-field approximation, when all the field-equations become linear. In this linear approximation the theory is equivalent to a quantized field theory of the usual type, derived from a Hamiltonian quadratic in the field variables, and describing noninteracting particles propagating freely through the vacuum. It differs from the standard formalism of Fierz and Pauli [Proc. Roy. Soc. London. Ser. A. 173, 211–232 (1939); these Rev. 1, 190] for particles of spin 2 only in the form of the supplementary conditions.

Aside from the mathematical development of the theory, some statements of a physical nature are made. For example, it is said that the charged particles observed in the beta-decay of nuclei are not electrons, but are particles with integer spin and the same mass and charge as the electron. Such particles are indeed predicted by the theory. But no attempt is made to discover whether this explanation of beta-decay is consistent with experimentally known facts.

F. J. Dyson (Ithaca, N. Y.).

Cini, M., e Radicati, L. A. Un principio variazionale per problemi dipendenti dal tempo. Nuovo Cimento (9) 7, 905–910 (1950).

The formalism of Feynman's theory [Physical Rev. (2) 76, 769–789 (1949); these Rev. 11, 765] is used to generalize a variational method developed by Schwinger and others [see, for instance, Kohn, ibid. 74, 1763–1772 (1948)] in such a way that it applies to time-dependent potentials. The transition amplitude from an initial to a final state is shown to be stationary with respect to small variations of the wave function for the final state. In the meantime, the same result

has been obtained by Lippmann and Schwinger [ibid. **79**, 469–480 (1950); these Rev. **12**, 570] in a different way.

E. Gora (Providence, R. I.).

Cini, Marcello. A variational principle for time-dependent problems. Physical Rev. (2) **80**, 300–301 (1950).

Consider the motion of a single Dirac electron in a given potential $A(x, t)$. Let the electron be in a state f at time t_1, and let g be any state orthogonal to f. Let α be the transition probability amplitude for finding the electron in the state g at a later time t_2. The potential acts on the electron transforming the initial wave-function $f(1)$ into $\psi(2)$ which obeys the integral equation $\psi(2)=f(2)-i\int K_+(2,1)A(1)\psi(1)d\tau_1$. Similarly, the potential acts on the final wave-function $g(2)$ to give the wave-function $\chi(2)$ which satisfies

$$\bar{\chi}(2)=\bar{g}(2)-i\int \bar{\chi}(1)A(1)K_+(1,2)d\tau_1.$$

The perturbed wave-functions ψ and χ will in practice not be known exactly. The purpose of a variational principle is to obtain a formula for α which is more accurate than the best available approximations to ψ and χ. Such a formula is

$$\alpha=XY/(Z+iW),\quad X=\int \bar{\chi}(2)A(2)f(2)d\tau_2,$$

$$Y=\int \bar{g}(2)A(2)\psi(2)d\tau_2,\quad Z=\int \bar{\chi}(2)A(2)\psi(2)d\tau_2,$$

$$W=\int\int \bar{\chi}(3)A(3)K_+(3,2)A(2)\psi(2)d\tau_2 d\tau_3.$$

If ψ and χ here differ from their correct values by arbitrary independent errors of order ϵ, the error in α is only of order ϵ^2.

F. J. Dyson (Ithaca, N. Y.).

Cini, M. Su alcune relazioni tra principi variazionali nel quadro delle formulazioni di Feynman e di Schwinger dell'elettrodinamica. Nuovo Cimento (9) **7**, 911–918 (1950).

The author has formulated a general variational principle for the transition probability amplitude of a system under a time-dependent perturbation [see the preceding review]. In this paper it is shown how this principle appears naturally both in the Feynman and in the Schwinger formulation of quantum electrodynamics. In the case of a time-independent perturbation the principle reduces to the well-known variational principle of Schwinger for scattering amplitudes [Lippmann and Schwinger, Physical Rev. (2) **79**, 469–480 (1950); these Rev. **12**, 570].

F. J. Dyson.

Steinwedel, Helmut. Zum Formalismus linearer Feldtheorien. Z. Naturforschung **6a**, 123–133 (1951).

Feynman's version of quantum electrodynamics [Physical Rev. (2) **76**, 749–759, 769–789 (1949); these Rev. **11**, 765] is based on a variational principle which, in the usual notation, is

$$\delta\left\{-\sum m_a c\int d\alpha+\tfrac{1}{2}\sum(e_a e_b/c)\int\int f(\Gamma)(d_\mu b^\mu)d\alpha d\beta\right\}=0$$

where $\Gamma=-(a_\mu-b_\mu)(a^\mu-b^\mu)$. When f is the Dirac delta function, Wheeler and Feynman [Rev. Modern Physics **21**, 425–433 (1949); these Rev. **11**, 293] have shown that this principle leads to classical electrodynamics. In quantum field theory the singularities of the delta function give rise to well-known divergences. By choosing more regular functions for f, Feynman, and also McManus [Proc. Roy. Soc. London. Ser. A. **195**, 323–336 (1948); these Rev. **10**, 664] are able to avoid the divergences. As long as f approximates a delta function sufficiently closely the theory is experimentally indistinguishable from Maxwell's. The present paper seeks to answer the following questions, in the "hyperbolic" case in which $f(\Gamma)$ is identically zero for a space-like interval: (1) To what field equations will a given form of f lead? (2) What form of f is implied by given field equations? (3) What form of f and what field equations are demanded by modifications of the Coulomb law? The main mathematical tool is the Hankel transform. The regularizing weight function $\rho(\eta)$ of Pauli and Villars [Rev. Modern Physics **21**, 434–444 (1949); these Rev. **11**, 301] appears as the Hankel transform of a function closely related to f. The conditions (1) $\int_0^\infty \eta^m\rho(\eta)d\eta=0$, $m=0, 1, \cdots, n$, which Pauli and Villars use for $n=1$, are regularity conditions on f and are shown to imply that the field equations will have higher (in general, infinite) order. When conditions (1) are of the form $\sum_\nu c_\nu \eta_\nu^m=0$, the author obtains a new proof of Thirring's result [Physical Rev. (2) **77**, 570 (1950)] that they are of order $2n+4$. The theory is applied to classical electrodynamics and to Bopp's theory [Ann. Physik (5) **42**, 573–608 (1943); these Rev. **8**, 124] which uses fourth order field equations. A brief discussion of Born's theory [Rev. Modern Physics **21**, 463–473 (1949)] which employs the infinite order field equation $e^{-a^2\Box}\Box A_\mu=-4\pi j_\mu$ suggests that it is not amenable to investigation by the methods of the present paper.

A. J. Coleman (Toronto, Ont.).

Markov, M. A. On nonlocalized fields. I. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **21**, 11–15 (1951). (Russian)

A form of nonlocalizable theory is proposed, in which the component $A_\mu(k)$ of the electromagnetic potential operator with propagation-vector k does not commute with the space-time coordinate x_ν. The commutation law $[A_\mu(k), x_\nu]=-ib^2k_\nu A_\mu(k)$ is suggested as a possibility, with b a universal constant with the dimension of a length. This theory has some similarity to the nonlocal formalism of H. Yukawa [Physical Rev. (2) **76**, 300–301 (1949)]. The consequences of the theory are not explored in detail. The author claims that his formalism gives an unambiguous and consistent definition of a wave-function describing the behaviour of interacting particles so long as these particles are well separated from each other, and in particular including an account of the asymptotic behaviour of the particles at infinity. He states also that the wave-function automatically vanishes when the particles are close together (in comparison with the natural unit of length b) thus forbidding a strictly local description of collision processes.

F. J. Dyson (Ithaca, N. Y.).

Rayski, J. Remarks on the non-local electrodynamics. Proc. Roy. Soc. London. Ser. A. **206**, 575–583 (1951).

Yukawa's nonlocal quantum theory and Born's reciprocity principle motivate the introduction of convergence factors into the usual series for the S-matrix. Charge density is treated as nonlocal, the electromagnetic field as local. This procedure eliminates infinities due to self-charge. However, the theory is not gauge invariant and so to remove photon self-energy and zero-point vacuum energy recourse is made to a realistic mixture of charged fermions and bosons.

A. J. Coleman (Toronto, Ont.).

Snow, George, and Snyder, Hartland S. On the self-energies of quantum field theory. Physical Rev. (2) **80**, 987–989 (1950).

The method described in previous papers [Snyder, same Rev. (2) **78**, 98–103 (1950); **79**, 520–525 (1950); these Rev. **11**, 632; **12**, 226] is here applied to a calculation of the self-energy of a photon, and to a calculation of the current associated with a mixture of zero-photon and one-photon states. These two quantities should, for physical reasons, both be zero. However, a certain linear combination of the two, when calculated to the order e^2, turns out to be a positive definite integral and cannot be made to vanish by the limiting process prescribed by the authors. This shows that the authors' method fails in its ostensible purpose of removing all divergence difficulties from quantum electrodynamics. The authors observe that, combining their method with the introduction of a supplementary boson field, they would still have some hope of carrying through their original program. *F. J. Dyson* (Ithaca, N. Y.).

Feynman, R. P. Mathematical formulation of the quantum theory of electromagnetic interaction. Physical Rev. (2) **80**, 440–457 (1950).

This paper gives the logical and rigorous theoretical foundation upon which the author's previously published radiation theory [same Rev. (2) **76**, 749–759, 769–789 (1949); these Rev. **11**, 765] is based. The method followed is simply to apply the author's Lagrangian formulation of quantum mechanics [Rev. Modern Physics **20**, 367–387 (1948); these Rev. **10**, 224] to the variables describing the electromagnetic field oscillators. Since the Lagrangian is a quadratic function of the oscillator amplitudes, the integration of the action integrals with respect to these amplitudes can always be carried out exactly. In this way the field variables become entirely eliminated from the theory, the effects of the electromagnetic field being represented by retarded interactions between charges and currents.

The central result of the paper is the following. Let any system of charges, each carrying the same charge e, be interacting with each other through the quantized electromagnetic field. Let each charge at the same time be subject to an external force generated by a classical 4-vector potential energy B_μ, a function of space and time. It is supposed that the charge e and the potentials B_μ can be independently varied. Let $T=T(e^2, B)$ be the transition probability amplitude for finding the system in a given final state at time t'', given the initial state at an earlier time t'. Then T satisfies the functional differential equation

$$\frac{dT(e^2, B)}{d(e^2)}=\tfrac{1}{2}i\int_{t'}^{t''}\int_{t'}^{t''}\frac{\delta^2T(e^2, B)}{\delta B_\mu(1)\delta B_\mu(2)}\delta_+(S^2_{12})d\tau_1 d\tau_2,$$

where the integration is over the two space-time points x_1 and x_2, and S_{12} is the invariant length of the vector (x_1-x_2). The amplitude $T(0, B)$ describes the behaviour of noninteracting particles in an external field, and can be computed exactly in all cases of interest. The differential equation thus determines $T(e^2, B)$ uniquely, given the boundary value at $e^2=0$. When the equation is solved by a series expansion in e^2, the result is the rules of calculation given in the author's radiation theory [loc. cit.].

In the final section of the paper, some additional rules are deduced which extend the results to all processes in which real photons are present in the initial or final state. Three appendices follow, each containing in condensed form a number of important ideas. Appendix A gives a formally covariant treatment of particles satisfying the Klein-Gordon equation, making use of proper-time as an independent variable; this theory has close connections with the proper-time formalisms of Y. Nambu [Progress Theoret. Physics **5**, 82–94 (1950); these Rev. **11**, 766] and of J. Schwinger [see the following review]. Appendix B discusses, in a general way, relations between real and virtual processes involving photons. Appendix C derives a wave-equation for $\Phi(e^2, B, x)$, the probability amplitude for finding a Dirac electron at the space-time point x_μ, given any initial conditions at any earlier time. The electron is supposed subject to an external potential energy B_μ as before, and is also supposed to interact with the quantized electromagnetic field through its charge e. The wave-equation is

$$(i\gamma_\mu(\partial/\partial x_\mu)-m)\Phi = \gamma_\mu B_\mu(x)\Phi+ie^2\gamma_\mu\int\delta_+(S^2_{x_1})(\delta_1\Phi/\delta B_\mu(1))d\tau_1.$$

This equation contains in compact form the modification introduced into the Dirac theory of the electron by the interaction of the electron with its own field.

The present paper is the first published account which gives an adequate idea of the scope and power of the Lagrangian formulation of quantum mechanics. In the hands of the author and of Schwinger [unpublished] the Lagrangian method has led directly to elegant solutions of a great variety of problems in quantum field theory, only a few of which are dealt with in the present paper. *F. J. Dyson.*

Schwinger, Julian. On gauge invariance and vacuum polarization. Physical Rev. (2) **82**, 664–679 (1951).

The manifestly Lorentz-invariant approach to quantum field theory that has proved so useful in the reduction of divergence difficulties is cast into a form that preserves gauge-invariance at all stages. This result is achieved by expressing all operators in terms of the field strengths except in so far as the properly gauge-dependent phase factors are concerned. The method is illustrated by the calculation of the polarization of the electron-positron vacuum by a given electromagnetic field. The Green's function for the Dirac field in the presence of the external field is an intermediate result of the calculation.

The success of the method rests on the fact that the relevant quantities (Green's function, energy operator, etc.) can be expressed as integrals over a proper time parameter. All necessary operations are then performed on the integrand as a function of proper time, and they all yield finite results, for the divergences can all be brought to reside in the proper time integration. Now the physical significance of a contribution (whether renormalization or real effect) can be decided before this integration is carried out and the undetectable quantities, which, in electrodynamics, include all the divergences, can be interpreted and removed by renormalization.

Specific results include an exact treatment of constant fields and of plane wave fields and a perturbation method that can be applied to arbitrarily varying fields. Previously proposed regularization methods [W. Pauli and F. Villars, Rev. Modern Physics **21**, 434–444 (1949); these Rev. **11**, 301] for handling divergent integrals are shown to be included in the proper-time technique. The two-photon disintegration of the spin zero neutral meson due to its interaction with the proton vacuum is so similar to quantum electrodynamic processes that it can be obtained from expressions derived initially for other purposes.

R. Karplus (Cambridge, Mass.).

Fukuda, Hiroshi, and Kinoshita, Toichiro. Ambiguities in quantized field theories. Progress Theoret. Physics 5, 1024–1032 (1950).

A further discussion of the problem which arises in quantized field theories when perturbation-theoretic calculations give results in conflict with certain formal identities which the theories should satisfy. [For a statement of the problem see Fukuda, Hayakawa, and Miyamoto, same journal 4, 347–357 (1949); Fukuda and Miyamoto, ibid. 5, 283–304 (1950).] In this paper it is shown that the formal identities lead to certain relations between divergent and semiconvergent integrals, which are in some cases inconsistent. No solution of the problem is offered. New light has recently been thrown by Schwinger on these questions [see the preceding review]. *F. J. Dyson* (Ithaca, N. Y.).

Källén, Gunnar. Formal integration of the equations of quantum theory in the Heisenberg representation. Ark. Fys. 2, 371–410 (1951).

In a previous paper [same vol., 187–194 (1950); these Rev. 12, 502] the author has developed a method of solving the differential equations of quantum electrodynamics, working all the time in the Heisenberg representation and not using the interaction representation. In this paper the formal solutions are examined in detail. In the introductory sections it is shown that the author's method of solution must always give results in agreement with those obtained by using the interaction representation. The main part of the paper (section 5) is devoted to an analytical reduction of some of the extremely complicated multiple commutators and integrals, which arise in the higher terms of the series expansions in which the solutions are obtained. From this analysis it is clear that the solutions are closely related to the expressions occurring in the series expansion of the S-matrix [the reviewer, Physical Rev. (2) 75, 1736–1755 (1949); these Rev. 11, 145]; in particular, the two expansions coincide when no real creation of particles can take place. At the end a short section shows how the author's method can be extended without difficulty to the study of fields with integer spin and of interactions containing derivatives. *F. J. Dyson* (Ithaca, N. Y.).

de Broglie, Louis. Sur la possibilité d'une structure complexe des particules de spin différent de ½. J. Phys. Radium (8) 12, 509–516 (1951).

With his method of fusion [cf. Théorie générale des particules à spin. Méthode de fusion, Gauthier-Villars, Paris, 1943], the author obtained equations for particles of arbitrary spin S, by regarding them as a "fusion" of $2S$ particles of spin ½. Here, he studies the possibility that fusion is not merely a mathematical trick but corresponds to physical reality. For this it is necessary to assume an interaction among the particles which is introduced along the lines of a paper of Fermi and Yang [Physical Rev. (2) 76, 1739–1743 (1949)]. In the case $S=1$ coordinates of the center of mass and relative coordinates of two particles of spin ½ are introduced permitting the author to conclude that the particle consists of an "external" particle described by a vector ($S=1$) and an "internal" particle described by a pseudoscalar ($S=0$), and that the probability density in q-space is not $|\psi|^2$ but $(1-v^2/c^2)^{\frac{1}{2}}|\psi|^2$. [Historical remark by reviewer: The so-called Proca equations might more properly be called the de Broglie-Proca equations since de Broglie used them two years before Proca. Cf. Une nouvelle conception de la lumière, Actualités Sci. Ind., no. 181, Hermann, Paris, 1934.] *A. J. Coleman.*

de Broglie, Louis. Schéma lagrangien de la théorie du champ soustractif. C. R. Acad. Sci. Paris 232, 1269–1272 (1951).

Let A^α, $\alpha=1, 2, 3, 4$, be the potentials of a vector field, and $F^{\alpha\beta}=(\partial A^\alpha/\partial x_\beta)-(\partial A^\beta/\partial x_\alpha)$. It is desired to make the field represent particles with two distinct characteristic masses k_1 and k_2. For this purpose the following Lagrangian is proposed

$$L=\frac{1}{4(k_1^2+k_2^2)}\frac{\partial F^{\alpha\beta}}{\partial x_\gamma}\frac{\partial F^{\alpha\beta}}{x_\gamma}+\tfrac{1}{4}F^{\alpha\beta}F^{\alpha\beta}+\frac{k_1^2k_2^2}{2(k_1^2+k_2^2)}A^\alpha A^\alpha.$$

It is claimed that this field will give rise to an automatic compensation of the divergences in quantum electrodynamics, as described in earlier papers by the author [Portugaliae Math. 8, 37–58 (1949); J. Phys. Radium 11, 481–489 (1950); these Rev. 11, 763; 12, 464]. *F. J. Dyson* (Ithaca, N. Y.).

Ferretti, B. Ancora sull'operatore $S(\sigma)$ di Dyson-Feynmann. Nuovo Cimento (9) 7, 783–785 (1950).

This paper is one of a series [same vol., 79–81, 375–377, 899–900 (1950); these Rev. 11, 568; see also the following review] which is concerned with the general problem of extending the reviewer's method of calculating the S-matrix in quantum electrodynamics [Physical Rev. (2) 75, 486–502, 1736–1755 (1949); these Rev. 10, 418; 11, 145] to systems involving bound states. The reviewer's method was restricted to pure scattering processes between free particles; the extension to bound states is both difficult and urgently necessary. The author introduces explicitly the notion of an "adiabatic switching on" of the electronic charge e, or more generally of a coupling constant λ. Thus $\lambda(t)$ is considered to be a function of the time t, varying very slowly from the value zero in the remote past to a given value at a given time. The S-matrix may be formally calculated with the varying $\lambda(t)$. However, if bound states are present this S-matrix will not tend to any limit as the rate of variation is made infinitely slow. A new operator is defined, which does tend to a finite limit S_e in these circumstances. The S_e is the S-matrix in a certain representation, different from the representation in which the previous calculations of the S-matrix have been made. The rules for calculating S_e are very briefly explained; further details will be published later. *F. J. Dyson* (Ithaca, N. Y.).

Ferretti, B. Sulla diagonalizzazione della hamiltoniana nella teoria dei campi d'onda e sulla teoria dei sistemi chiusi. I. Nuovo Cimento (9) 8, 108–131 (1951).

Consider a physical system defined by the Schrödinger equation $i\hbar(\partial\phi/\partial t)=(H_0+\lambda V)\phi$. Here H_0 is an "unperturbed Hamiltonian," V is a perturbing interaction, and λ is a coupling constant. In particular, quantized field theories such as quantum electrodynamics are systems of this type. In order to determine the stationary states of such a system, it is convenient to look for a unitary operator Π which diagonalizes the complete Hamiltonian, i.e., an operator Π such that $E(\lambda)=\Pi^{-1}(H_0+\lambda V)\Pi$ is diagonal in the representation in which H_0 is diagonal. Such an operator is defined formally by the infinite product

$$\Pi(t)=\prod_{n=0}^{\infty}\left[1-(i/\hbar)\int_{t_{n+1}}^{t_n}\lambda(t')V(t')dt'\right],$$

where $t_0=t, t_1, t_2, \cdots$ are a sequence of times extending from t to $-\infty$ and becoming in the limit infinitely close to one another. The coupling constant λ is replaced by a function

of time $\lambda(t')$, which is supposed to increase very slowly from the value zero at $t'=-\infty$ to the value λ at $t'=t$. This "adiabatic switching on of the interaction" is essential in order to make $\Pi(t)$ a well-defined operator.

Now if the product $\Pi(t)$ is formally expanded into a power series in the interaction, the result is

$$S(t)=1-(i/\hbar)\int_{-\infty}^{t}\lambda(t')V(t')dt' -(1/\hbar^2)\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt''\lambda(t')V(t')\lambda(t'')V(t'')+\cdots.$$

The methods of removal of divergences from quantized field theories by renormalization of constants are always applied to the series $S(t)$ and not to the product $\Pi(t)$. Therefore if the renormalization method is to be applied in its present form to the study of bound states, we must be able to make use of the identity $\Pi(t)=S(t)$. The present paper is concerned with the question, under what conditions this identity is valid. It is shown quite generally that the identity is not valid when the total Hamiltonian $(H_0+\lambda V)$ has discrete eigenvalues. This is true even when the eigenvalues in question can be calculated by a perturbation expansion in powers of λ. There is also an illuminating discussion of the series $S(t)$, in which it is shown that this series actually diverges in the case of discrete eigenvalues. In the case of continuous eigenvalues, when the series $S(t)$ is meaningful, it is necessary to take account explicitly of the time-variation of $\lambda(t')$ when calculating the operator $E(\lambda)$.

The importance of the results is that they show the existing renormalization technique to be entirely inadequate for a correct treatment of bound states. The author promises in later papers to supply a new method of carrying through the renormalization program which shall overcome these difficulties. *F. J. Dyson* (Ithaca, N. Y..)

Nambu, Yôichirô. Force potentials in quantum field theory. Progress Theoret. Physics **5**, 614–633 (1950).

If two particles both interact with a quantized field ϕ_λ (λ representing tensor or isotopic spin indices), a force between the two particles results. One of the central problems in quantum field theory is to determine this force, given the character of the field ϕ_λ. In the classical and nonrelativistic approximation, the force is derivable from a potential energy of interaction which can easily be calculated. When relativistic and quantum-mechanical recoil effects are taken into account, it becomes difficult even to formulate the problem of calculating the force in a clear and unambiguous way.

The present paper is an illuminating discussion of the general questions, under what conditions a potential energy of interaction can be defined, and to what extent this potential gives a correct description of the interaction. The case where ϕ_λ is the electromagnetic field is studied in detail. A general form of the potential is given which is valid in the nonrelativistic weak-coupling approximation. The result shows that the corrections which should be made to the potential directly derivable from classical theory are essentially dependent on the nature of the interaction. Those interactions which contain only classical variables give in the quantum theory also potentials not much different from classical ones. On the other hand, if the interaction contains quantum variables, such as the Dirac spin and isotopic spin matrices, the higher order corrections in general affect the classical potential radically.

In the case of the electromagnetic field, the interaction potential is calculated up to terms of order e^4. The result is simply the Breit interaction, omitting the matrix elements giving transitions to negative energy states which lead to clearly erroneous results when taken seriously [Breit, Physical Rev. (2) **36**, 383–397 (1930); **39**, 616–624 (1932)]. There remain no net fourth order effects of detectable magnitude.

In conclusion the author states, "The concept of potential energy cannot enjoy a wide and practical extension much beyond the classical and nonrelativistic form." He then mentions a wave equation of a new type, namely,

$$\left(\gamma_\lambda\frac{\partial}{\partial x_\lambda}-\kappa\right)_1\left(\gamma_\lambda\frac{\partial}{\partial x_\lambda}-\kappa\right)_2\psi=-\frac{e^2}{2}(\gamma_\nu)_1(\gamma_\nu)_2D_F(12)\psi,$$

which he suggests as the starting-point for a more correct and relativistic treatment of the two-body problem. The same equation has independently been proposed by Bethe and Salpeter, and by Schwinger [unpublished]. The precise meaning of this equation has not yet been made clear by anybody, but the author is certainly right in attaching importance to it. *F. J. Dyson* (Ithaca, N. Y.).

Thermodynamics, Statistical Mechanics

Jancel, Raymond. Le deuxième principe de la thermodynamique et la mécanique ondulatoire. Revue Sci. **88**, 160–177 (1950).

The various attempts to find a basis for quantum statistical mechanics are analyzed. It is shown that most of the hypotheses that have been made are in contradiction with elementary properties of the Schrödinger equation. A resolution of the problem using the work of Davydov [Acad. Sci. USSR. J. Phys. **11**, 33–43 (1947); these Rev. **9**, 402] is proposed. This solution makes decisive use of the quantum theory of measurement. *K. M. Case.*

Mott-Smith, H. M. The solution of the Boltzmann equation for a shock wave. Physical Rev. (2) **82**, 885–892 (1951).

Although there have been recently several detailed calculations of the structure and "thickness" of the shock wave by using the Navier-Stokes equations of gas dynamics, the fact remains that at Mach numbers greater than two, the thickness is only of the order of a few mean free paths of the molecules and therefore the usual continuum concept must break down. A fresh start is thus necessary from the Boltzmann equation. This paper is the first published study with the correct approach, although the more general problem of large gradients was analysed by H. Grad [Comm. Pure Appl. Math. **2**, 331–407 (1949); these Rev. **11**, 473] without numerical results. The author approximates the molecular distribution function at any space point x in the shock by the sum of two terms. Each term is a Maxwellian distribution corresponding to the equilibrium states of the gas far ahead and far downstream of the shock. The relative weight of these two terms, or the number of molecules in either one of the two Maxwellian distributions is an unknown function of the space point x, to be determined by substituting the assumed distribution into the Boltzmann equation. Actually, the problem is solved approximately by requiring the second or the third moment of the distribution function with respect to the x-component of the molecular velocity to

satisfy the Boltzmann equation. Either condition leads to a thickness much larger than that of the gas dynamic theory, and this is in agreement with recent measurements of Cowan and Hornig [J. Chem. Physics **18**, 1008–1018 (1950)] and of Greene, Cowan, and Hornig [ibid. **19**, 427–434 (1951)]. *H. Tsien* (Pasadena, Calif.).

Van Hove, L. Un problème d'intégrations posé par la mécanique statistique. Bull. Soc. Math. Belgique **2** (1948–1949), 49–55 (1950).

This is a discussion of the partition function of a one-dimensional gas. This function is evaluated by a combinatory method which employs the asymptotic formula for the "partitio numerorum" of Hardy and Ramanujan. Similar results have been obtained by the use of Laplace transforms [cf. Takahasi, Proc. Phys.-Math. Soc. Japan (3) **24**, 60–62 (1942); these Rev. **7**, 540]. Unfortunately neither of the methods seems to be useful for the investigation of a three-dimensional gas or liquid. *E. W. Montroll.*

Cottrell, T. L., and Paterson, S. The virial theorem in quantum mechanics. Philos. Mag. (7) **42**, 391–395 (1951).

This is an extension of the quantum mechanical virial theorem for particles which are localized in a box. If E is the energy level of a system of interacting particles the usual virial theorem

$$-2\langle T\rangle_{\mathrm{av}}=-\langle\sum\mathbf{r}_i\cdot\nabla_iV\rangle_{\mathrm{av}}$$

must be replaced by

$$-2\langle T\rangle_{\mathrm{av}}=-\langle\sum\mathbf{r}_i\cdot\nabla_iV\rangle_{\mathrm{av}}+a\partial E/\partial a,$$

where the averages are time averages, T = kinetic energy of system, V is potential energy of interaction of particle, and A is a parameter related to the dimension of the box. If V is interpreted to be the potential of external as well as mutual interaction forces, the first expression can easily be shown to be valid again unless the wall potential is infinite. The present controversy between H. S. Green [Physica **15**, 882–890 (1949); J. Chem. Phys. **18**, 1123–1124 (1950); these Rev. **11**, 634; **12**, 576], de Boer [Physica **15**, 843–848 (1949); these Rev. **11**, 634], and others involves just this point when the wall potentials are infinite.

E. W. Montroll (College Park, Md.).

Rushbrooke, G. S., and Scoins, H. I. On virial coefficients and the Born-Green theory of fluids. Philos. Mag. (7) **42**, 582–593 (1951).

The behavior of the virial coefficients which one obtains by using the Kirkwood "superposition" approximation in the Born-Green theory of fluids is studied. The second and third virial coefficients are given exactly, but the fourth is found to be erroneous. It is also shown that if the Born-Green equations are linearized the second and third coefficients remain unchanged and correct, while the fourth is changed and remains incorrect. Numerical values are given for the case of hard spheres, where it is found that the non-linear equation gives too small a fourth virial coefficient, while the linearized equation gives a result too large. Finally, the use of this approximation in the theory of binary mixtures is discussed. It is found that the resulting nonlinear equations are mathematically inconsistent and only become compatable when linearized. *J. M. Luttinger.*

Kirkwood, John G., and Buff, Frank P. The statistical mechanical theory of solutions. I. J. Chem. Phys. **19**, 774–777 (1951).

The formal theory of the statistical mechanics of solutions is developed. It is shown how one may calculate various quantities of interest such as the derivatives of the chemical potentials and osmotic pressure with respect to concentrations, the partial molar volumes, and compressibility. Using the grand canonical ensemble these quantities are related to composition fluctuations, which are in turn related to certain integrals over the radial distribution functions of the various pairs present. The case of two components is treated in detail, and for this case expansions in terms of variables related to the concentrations are given.

J. M. Luttinger (Madison, Wis.).

Guggenheim, E. A., and McGlashan, M. L. Statistical mechanics of regular mixtures. Proc. Roy. Soc. London. Ser. A. **206**, 335–353 (1951).

The problem of finding the partition function of a regular mixture or, equivalently, of the Ising model of a ferromagnet in an external magnetic field, is considered. The problem has been solved exactly for the one-dimensional case, but never in two or three dimensions. Here a new method of approximation is suggested in which either pairs, triplets or quadruplets of neighboring sites are treated as independent units. The pair approximation is equivalent to the older and better known Bethe-Peierls and quasi-chemical methods. The triplet and quadruplet approximations on the other hand might be thought of as an improvement on these. It is shown by direct calculation that pair, triplet and quadruplet units all lead to very closely the same result, which is taken as an indication of the reliability of the older approximation.

J. M. Luttinger (Madison, Wis.).

Fisher, E. Partition functions of cubic lattices. J. Chem. Phys. **19**, 632–640 (1951).

The partition function for the Born-von Kármán model of a simple cubic elastic lattice is obtained approximately from the energy function given by Blackman [Proc. Roy. Soc. London. Ser. A. **159**, 416–431 (1937)]. The resulting expression for the partition function contains in an essential manner an expression in the form $\ln(r^{-1}\sinh r)$. Since $r^{-1}\sinh r=\prod_{n=1}^{\infty}(1+(r/n\pi)^2)$, the partition function can be expressed as a series of terms depending on n. In specific examples, the author shows that a few of these terms (i.e. up to $n=3$ or 4) will give adequate accuracy. The expression obtained by the author involves a triple integral to which the author applies a series integration by a Taylor series, taken at strategically placed values. A brief sketch of the convergence considerations involved is given. In order to handle the complicated expressions involved, one must make a careful choice of the point at which the Taylor's series expansion is taken, but since the number of available points is limited, the convergence becomes uncertain at low temperatures. For the body centered cubic, an analogous discussion is given based on the work of Montroll and Peaslee and for the face centered cubic, based on the work of R. B. Leighton. *F. J. Murray* (New York, N. Y.).

DECEMBER ISSUE IS AN INDEX WHICH HAS
BEEN PHOTOGRAPHED AT THE BEGINNING
OF THE VOLUME(S).